INDUCING SCHEMES FOR MULTI-DIMENSIONAL PIECEWISE EXPANDING MAPS

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ABSTRACT. We construct inducing schemes for general multi-dimensional piecewise expanding maps where the base transformation is Gibbs-Markov and the return times have exponential tails. Such structures are a crucial tool in proving statistical properties of dynamical systems with some hyperbolicity. As an application we check the conditions for the first return map of a class of multi-dimensional non-Markov, non-conformal intermittent maps.

1. INTRODUCTION

Statistical properties of chaotic dynamical systems have been a subject of interest for mathematicians and physicists in the past several decades. While such properties are better understood for uniformly hyperbolic systems, the same cannot be said about systems with non-uniform hyperbolicity. The reason is that there are many mechanisms for non-uniform behaviour (e.g. intermittency, existence of critical points or singularities, etc.) and usually they are mixed with regions (or periods) of uniformly hyperbolic behaviour. To treat this difficulty Young [8, 9] proposed an abstract framework to study such systems. She showed that if the system admits a certain structure, called a Young tower, then statistical properties such as rates of decay of correlations can be deduced using the analogy to Markov chains. Since then many other statistical properties have been studied assuming the existence of such structures. However, constructing such structures for various systems is not easy and requires a good understanding of the nature of non-uniformity of hyperbolicity. Even then, it is usually done in a case by case basis.

The purpose of this article is to obtain such structures for general multidimensional expanding maps with discontinuities. One of the main applications is proving statistical properties for other non-uniformly hyperbolic maps. This is because often through some kind of initial inducing (e.g. first return to a subset away from non-uniformity) one can replace the non-uniformity with the presence of discontinuities; then the results of this paper can be applied to upgrade the initial inducing scheme to a superior one from which virtually every statistical property of interest can be obtained for the original system. Such properties include existence and properties of absolutely continuous invariant measures, decay of correlations, central limit theorem, large deviations, Berry-Esseen theorem, almost sure invariance principle, law of iterated logarithm, etc. As an example, this procedure is applied to a class of multi-dimensional intermittent maps introduced in [4], see section 4.

Inducing schemes have been constructed in the past, but to my knowledge, none of them cover the case of multidimensional piecewise expanding maps. Almost all of them pertain to systems with one-dimensional expanding direction (e.g. the original [8]) with the exception of [3], which applies to multidimensional invertible

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maps with discontinuities. Even though the ideas of [3] have influenced our work, it should be evident that the results of this paper cannot be obtained from [3] because piecewise expanding maps are non-invertible. Furthermore, our assumptions on the discontinuities are weaker than those of [3], which may provide some insight in generalizing [3] to cover multidimensional billiards. We refer the reader to the survey [6] for the relevance of piecewise expanding maps in providing insights into the problems surrounding the study of multi-dimensional billiards. Other systems to which the results of the current paper could possibly be applied (in my opinion) are variations on "Hu-Vaienti" maps [5] and "Viana" maps [7, 1]. This shall be subject of future work.

Finally, we would like to mention that there is a vast (50+ years of) literature on piecewise expanding maps including very important advances. But, our intention is not to survey everything that is known about such systems. The focus of this paper is obtaining inducing schemes in the multidimensional setting and as pointed out earlier there is very little previous work in this regard.

The paper is organized as follows. In Section 2 we present the setting and assumptions on the class of dynamical systems we consider. In Section 3 we present the statements of our main results involving the inducing schemes. In section Section 4 we describe a family of examples satisfying our assumptions. Then we proceed to the proofs. Section 5 describes the notion of standard families [3], which are then used in Section 6 to prove supplementary lemmas needed for the proof of the main theorems. Finally, we present the proofs of the main results.

2. Setting and assumptions

Consider $\mathbb{R}^d$ endowed with the Euclidean metric $\mathbf{d}$ and the Lebesgue measure $\mathbf{m}$. Let $X \in \mathbb{R}^d$ be a bounded, Borel measurable subset such that

$$\sup_{\varepsilon>0} \frac{\mathbf{m}(\partial_\varepsilon X)}{\varepsilon} < \infty, \tag{2.1}$$

where $\partial X = \operatorname{cl} X \cap \operatorname{cl}(\mathbb{R}^d \setminus X)$ is the topological boundary of X in $\mathbb{R}^d$ and $\partial_\varepsilon X = \{x \in X : \mathbf{d}(x, \partial X) < \varepsilon\}$. We consider a *non-singular piecewise invertible map* T on X with respect to the countable partition $\mathcal{P} = \{O_h\}_{h\in\mathcal{H}}$ of open subsets of X. This means that $\mathbf{m}(X \setminus \bigcup_{h\in\mathcal{H}} O_h) = 0$ and the restrictions $T : O_h \to T(O_h)$ and their inverses are non-singular (i.e. $\forall h \in \mathcal{H}$, $(T|_{O_h})_*(\mathbf{m}|_{O_h})$ is equivalent to $\mathbf{m}|_{TO_h}$) homeomorphisms of O_h onto $T(O_h)$. It is notationally convenient to use $h : TO_h \to O_h$ to also denote an inverse branch of T and use $\mathcal{H}$ to denote the set of inverse branches of T. Accordingly, we denote the set of inverse branches of T^n, $n \in \mathbb{N}$, by $\mathcal{H}^n$ and the corresponding partition by $\mathcal{P}^n$. We write Jh for the Radon-Nikodym derivative $d(\mathbf{m} \circ h)/d\mathbf{m}$. We make the following assumptions on our dynamical system.

Remark 1. Note that X is contained in $\operatorname{cl} X$ (closure of X), which is the disjoint union of *int* X (interior of X) and ∂X. Moreover, $\mathbf{m}(\partial X) = 0$ because of (2.1). So, we can restrict our dynamics to *int* X, which is an open set. So, without loss of generality, we may assume that X is an open subset of $\mathbb{R}^d$. Consequently, $I \subset X$ is open in X if and only if it is open in $\mathbb{R}^d$.

(1) Uniform expansion: For every $h \in \mathcal{H}$ and $\varepsilon > 0$, denote

$$\Lambda_h(\varepsilon) = \sup_{\{x,y\in T(O_h)\,:\mathbf{d}(x,y)\le\varepsilon\}} \frac{\mathbf{d}(h(x), h(y))}{\mathbf{d}(x, y)}.$$

There exist $\varepsilon_1 > 0$ and $\Lambda \in (0, 1)$ such that for every $h \in \mathcal{H}$, $\Lambda_h(\varepsilon_1) \le \Lambda < 1$. Set $\Lambda_h := \Lambda_h(\varepsilon_1)$. Note that for $h \in \mathcal{H}^n$, we can define Λ_h using T^n and it is easy to verify that for all $h \in \mathcal{H}^n$, $\Lambda_h(\varepsilon_1) \le \Lambda^n < 1$.

(2) Bounded distortion: There exist $\alpha \in (0,1]$, $\tilde{D} \geq 0$ such that $\forall h \in \mathcal{H}$, $\forall x, y \in T(O_h)$

$$Jh(x) \leq e^{\tilde{D}\mathbf{d}(x,y)^\alpha} Jh(y). \tag{2.2}$$

Let $D = \tilde{D}/(1 - \Lambda^\alpha)$. As a consequence of uniform expansion, (2.2) holds for $h \in \mathcal{H}^n$ uniformly for all $n \in \mathbb{N}$ with D instead of $\tilde{D}$.

(3) Controlled complexity: There exist $n_0 \in \mathbb{N}$, $\varepsilon_2 > 0$ and $0 \leq \sigma < \Lambda^{-n_0} - 1$ such that for every open set I, $\operatorname{diam} I \leq \varepsilon_2$, for every $\varepsilon < \varepsilon_2$,

$$\sum_{\{h \in \mathcal{H}^{n_0},\, \mathbf{m}(I \cap O_h) > 0\}} \frac{\mathbf{m}(h(\partial_\varepsilon T^{n_0}(I \cap O_h)) \setminus \partial_{\Lambda^{n_0}\varepsilon} I)}{\mathbf{m}(\partial_{\Lambda^{n_0}\varepsilon} I)} \leq \sigma < \Lambda^{-n_0} - 1. \tag{2.3}$$

Moreover, there exists a constant $\bar{C} < \infty$ such that for every integer $1 \leq r < n_0$, for every $\varepsilon < \varepsilon_2$,

$$\sum_{\{h \in \mathcal{H}^{r},\, \mathbf{m}(I \cap O_h) > 0\}} \frac{\mathbf{m}(h(\partial_\varepsilon T^{r}(I \cap O_h)) \setminus \partial_{\Lambda^{r}\varepsilon} I)}{\mathbf{m}(\partial_{\Lambda^{r}\varepsilon} I)} \leq \bar{C}. \tag{2.4}$$

We refer to the expression on the left-hand side of (2.3) as the *complexity expression*.

Remark 2. Often one can check (2.3) for $n_0 = 1$ in which case there is no need to check (2.4).

Remark 3. Suppose $h \in \mathcal{H}$ and $T|_{O_h} : O_h \to TO_h$ has an extension $\bar{T}_h : \operatorname{cl} O_h \to \operatorname{cl} TO_h$ that is invertible, its inverse $\bar{h}$ satisfies condition **(1)** and $\partial(TO_h) \subset \bar{T}_h(\partial O_h)$. Then if $A \subset O_h$, we have $\forall \varepsilon < \varepsilon_1$,

$$\begin{aligned} h(\partial_\varepsilon TA) = \bar{h}(\partial_\varepsilon TA) &= \bar{h}\{y \in T(A) : \mathbf{d}(y, \partial(TA) < \varepsilon\} \\ &\subset \{x \in A : \mathbf{d}(Tx, \bar{T}_h(\partial A)) < \varepsilon\} \\ &\subset \{x \in A : \mathbf{d}(x, \bar{h}(\partial(TA))) < \Lambda_h \varepsilon\} \\ &\subset \{x \in A : \mathbf{d}(x, \partial A) < \Lambda_h \varepsilon\} = \partial_{\Lambda_h \varepsilon}(A). \end{aligned}$$

This is a simple but useful fact to keep in mind when checking (2.3).

Fix

$$a_0 > D/(1 - \Lambda^\alpha) = \tilde{D}/(1 - \Lambda^\alpha)^2, \qquad \varepsilon_0 = \min\{\varepsilon_1, \varepsilon_2\}. \tag{2.5}$$

Remark 4. For technical reasons, we also require that ε_0 is sufficiently small that $\Lambda^{n_0}(1 + e^{a_0 \varepsilon_0^\alpha} \sigma) < 1$.

3. Statement of the main results

Theorem 1. *Suppose $T : X \circlearrowleft$ satisfies hypotheses **(1)**-**(3)**. There exist a countable partition $\mathcal{P}'$ of X into open sets (mod 0) and a function $\tau : X \to \mathbb{Z}^+$ constant on elements of $\mathcal{P}'$ such that*

(a) The map $G = T^\tau : X \circlearrowleft$ is a Gibbs-Markov map with finitely many images.
(b) $\mathbf{m}(\tau > n) \leq const \cdot \kappa^n$ for some $\kappa \in (0,1)$.

Remark 5. By a Gibbs-Markov map we mean a piecewise expanding map having the Markov property and uniform expansion and bounded distortion. By finitely many images we mean $\{G(P)\}_{P \in \mathcal{P}'}$ is finite.

The above induced map can in fact be upgraded to a full-branched Gibbs-Markov map.

Corollary 1. *Suppose $T : X \circlearrowleft$ satisfies hypotheses **(1)**-**(3)**. There exist an open set $Z \subset X$ and a refinement $\mathcal{P}''$ of the partition $\mathcal{P}'$ into open sets (mod 0) such that Z is a union of elements of $\mathcal{P}''$ and there exists a map $\tilde{\tau} : Z \to \mathbb{Z}^+$ constant on elements of $\mathcal{P}''$ such that*

(a) *The map* $\tilde{G} = T^{\tilde{\tau}} : Z \circlearrowleft$ *is a full-branched Gibbs-Markov map.*[1]
(b) $\mathbf{m}(\tilde{\tau} > n) \leq const \cdot \tilde{\kappa}^n$ *for some* $\tilde{\kappa} \in (0, 1)$.

The next theorem provides an inducing scheme with the additional property that the gcd of the return times is equal to 1.

Definition 1. We say that $Z \subset X$ has a *nice boundary* if there exists a constant $C_Z > 0$ such that $\forall \varepsilon \geq 0$, $\mathbf{m}(\partial_\varepsilon Z) \leq C_Z \varepsilon$ and for every open set $I \subset X$ containing Z,

$$\mathbf{m}(\partial_\varepsilon (I \setminus \mathrm{cl}\, Z) \setminus \partial_\varepsilon I) \leq C_Z \mathbf{m}(\partial_\varepsilon I).$$

Remark 6. Many geometric shapes have nice boundaries. For example, sets with piecewise smooth boundaries (no cusps) including rectangles and balls. The empty set $Z = \varnothing$ also has a nice boundary with $C_Z = 0$.

Definition 2. We say that $Z \subset X$ is *fully recurrent* (at times $\{n_j\}_{j=1}^K$) if there exist $K \geq 2$ positive integers $\{n_j\}_{j=1}^K$ such that $\gcd\{n_j\}_{j=1}^K = 1$ and the following holds: There exist inverse branches $h_{n_1} \in \mathcal{H}^{n_1}$, $h_{n_2} \in \mathcal{H}^{n_2}$, ..., $h_{n_K} \in \mathcal{H}^{n_K}$ such that $O_{h_{n_1}}, \ldots, O_{h_{n_K}}$ are pairwise disjoint and for every $j = 1, \ldots, K$,

$$T^{n_j}(O_{h_{n_j}} \cap Z) \supset Z.$$

Theorem 2. *Suppose* $T : X \circlearrowleft$ *satisfies hypotheses* ***(1)-(3)****. In addition, suppose that for every* $\delta > 0$ *there exists* $Z \subset X$ *of* $\mathrm{diam}\, Z \leq \delta$ *that is fully recurrent and has a nice boundary. Then, there exist* δ' *such that* $\forall \delta \leq \delta'$ *there exist a refinement* $\mathcal{P}''$ *of the partition* $\mathcal{P} \wedge \{Z\}$ *into open sets (mod 0) such that* $Z = Z(\delta)$ *is a union of elements of* $\mathcal{P}''$ *and there exists a map* $\tilde{\tau} : Z \to \mathbb{Z}^+$ *constant on elements of* $\mathcal{P}''$ *such that*
(a) *The map* $\tilde{G} = T^{\tilde{\tau}} : Z \circlearrowleft$ *is a full-branched Gibbs-Markov map.*
(b) $\gcd\{n \geq 1 : \mathbf{m}(\{\tilde{\tau} = n\}) > 0\} = 1$.
(c) $\mathbf{m}(\tilde{\tau} > n) \leq const \cdot \tilde{\kappa}^n$ *for some* $\tilde{\kappa} \in (0, 1)$.

The following theorem provides the same conclusions but under a different assumption than full-recurrence.

Theorem 3. *Suppose* $T : X \circlearrowleft$ *satisfies hypotheses* ***(1)-(3)****. In addition, suppose that for every* $\delta > 0$ *there exist* $Z \subset Z' \subset X$ *such that* Z *has a nice boundary,* $\mathrm{diam}\, Z' \leq \delta$*,* $\mathbf{m}(Z') > \mathbf{m}(Z)$ *and there exists* $h \in \mathcal{H}$ *such that* $O_h \subset Z$ *and* $TO_h \supset Z'$. *Then, there exist* δ' *such that* $\forall \delta \leq \delta'$ *there exist a refinement* $\mathcal{P}''$ *of the partition* $\mathcal{P} \wedge \{Z\}$ *into open sets (mod 0) such that* $Z = Z(\delta)$ *is a union of elements of* $\mathcal{P}''$ *and there exists a map* $\tilde{\tau} : Z \to \mathbb{Z}^+$ *constant on elements of* $\mathcal{P}''$ *such that*
(a) *The map* $\tilde{G} = T^{\tilde{\tau}} : Z \circlearrowleft$ *is a full-branched Gibbs-Markov map.*
(b) $\mathbf{m}(O_h \cap \{\tilde{\tau} = 1\}) > 0$.
(c) $\mathbf{m}(\tilde{\tau} > n) \leq const \cdot \tilde{\kappa}^n$ *for some* $\tilde{\kappa} \in (0, 1)$.

Remark 7. The above theorem is useful essentially when T has infinitely many branches, but note that one can either artificially introduce more branches by chopping up the original ones, or consider the first return map of T to some subset and apply the result to the new map.

4. Examples

Consider the family of maps $f : [0, 1] \times \mathbb{T} \circlearrowleft$ with $f(x, \theta) = (f_1(x, \theta), f_2(\theta))$, introduced in [4], where

$$f_1(x, \theta) = \begin{cases} x(1 + x^\gamma u(x, \theta)), & 0 \leq x \leq \frac{3}{4} \\ 4x - 3, & \frac{3}{4} < x \leq 1 \end{cases}, \qquad f_2(\theta) = 4\theta \bmod 1, \tag{4.1}$$

[1] Full-branched in the sense that $\tilde{G}(P) = Z$ for every $P \in \mathcal{P}''$.

and where, $\gamma > 0$ and $u : [0, \frac{3}{4}] \times \mathbb{T} \to (0, \infty)$ is a positive $\mathcal{C}^2$ function satisfying $u(0, \theta) \equiv c > 0$. Implicitly, it is also assumed that $x(1 + x^\gamma u(x, \theta)) \leq 1$ for all $x \in [0, \frac{3}{4}]$, $\theta \in \mathbb{T}$. It is also assumed that $|(Df)_{(x,\theta)} v| \geq |v|$ for all $(x, \theta) \in [0, \frac{3}{4}] \times \mathbb{T}$, $v \in \mathbb{R}^2$.

Finding directly a first return map for f which is uniformly expanding (with bounded distortion) and also Markov is nearly impossible. However, simply considering the first return map to a region away from the intermittent behaviour produces a piecewise expanding map, which with some work can be shown to satisfy the hypotheses of Section 2. Indeed, we can show that f has a first return map T that satisfies all the conditions of Theorem 3. It follows that T admits an inducing scheme with exponential tails. Then it is standard to show that f itself admits a suitable inducing scheme with estimates on the tail of the return times from which various statistical properties can be deduced (for f).

All of the conditions of Theorem 3 are essentially verified for a first return map of f in [4]. Indeed, following [4], we can define $M_i = f([0, \frac{3}{4}] \times [\frac{i}{4}, \frac{i+1}{4}])$, $i = 0, 1, 2, 3$. Then $M = \bigcup_{i=0}^{3} M_i$ is an invariant set for f and $f(M) = M$. We consider the first returns to the set $X = ([\frac{3}{4}, 1] \times \mathbb{T}) \cap M$. Let $\varphi : X \to \mathbb{Z}^+$ be the first return time, with first return map $T = f^\varphi : X \to X$. Hypotheses **(1)-(3)** are the first three statements of [4, Theorem A.1]. The last statement of [4, Theorem A.1] also provides the sets Z and Z' with the required properties. Note that the Z chosen in [4, Theorem A.1] is simply a rectangle so it has a nice boundary. Moreover, this rectangle contains infinitely many partition elements O_h of T, whose images under T individually cover nearly all of X, so item (b) of Theorem 3 holds on each one of them.

5. Toolbox

5.1. **Standard families.** For $\alpha \in (0, 1)$, and a function $\rho : I \to \mathbb{R}^+ := (0, \infty)$, $I \subset X$ define

$$H(\rho) := H_\alpha(\rho) = \sup_{x,y \in I} \frac{|\ln \rho(x) - \ln \rho(y)|}{\mathbf{d}(x, y)^\alpha}. \tag{5.1}$$

Definition 3 (Standard pair)**.** An (a, ε_0)–*standard pair* is a pair (I, ρ) consisting of an open set $I \subset X$ and a function $\rho : I \to \mathbb{R}^+$ such that $\operatorname{diam} I \leq \varepsilon_0$, $\int_I \rho = 1$ and

$$H(\rho) \leq a. \tag{5.2}$$

Remark 8 (Notation). All integrals where the measure is not indicated are with respect to the underlying measure $\mathbf{m}$.

Definition 4 (Standard family)**.** An (a, ε_0)–*standard family* $\mathcal{G}$ is a set of (a, ε_0)–standard pairs $\{(I_j, \rho_j)\}_{j \in \mathcal{J}}$ and an associated measure $w_\mathcal{G}$ on a countable set $\mathcal{J}$. The *total weight* of a standard family is denoted $|\mathcal{G}| := \sum_{j \in \mathcal{J}} w_j$. We say that $\mathcal{G}$ is an (a, ε_0, B)–*proper* standard family if in addition there exists a constant $B > 0$ such that,

$$|\partial_\varepsilon \mathcal{G}| := \sum_{j \in \mathcal{J}} w_\mathcal{G}(j) \int_{\partial_\varepsilon I_j} \rho_j \leq B|\mathcal{G}|\varepsilon, \text{ for all } \varepsilon < \varepsilon_0. \tag{5.3}$$

If $w_\mathcal{G}$ is a probability measure on $\mathcal{J}$, then $\mathcal{G}$ is called a *probability standard family.* Note that every (a, ε_0)–standard family induces an absolutely continuous measure on X with the density $\rho_\mathcal{G} := \sum_{j \in \mathcal{J}} w_j \rho_j \mathbb{1}_{I_j}$. We say that two standard families $\mathcal{G}$ and $\tilde{\mathcal{G}}$ are *equivalent* if $\rho_\mathcal{G} = \rho_{\tilde{\mathcal{G}}}$.

Next we define what we mean by an iterate of a standard family. Given an (a, ε_0)–standard family $\mathcal{G}$, we define an n-th iterate of $\mathcal{G}$ as follows.

Definition 5 (Iteration)**.** Let $\mathcal{G}$ be an (a, ε_0)–standard family with index set $\mathcal{J}$ and weight $w_{\mathcal{G}}$. For $(j, h) \in \mathcal{J} \times \mathcal{H}^n$ such that $\operatorname{diam} T^n(I_j \cap O_h) > \varepsilon_0$ and for an open set $V_* \subset T^n(I_j \cap O_h)$, $\operatorname{diam} V_* \leq \varepsilon_0/(4d^{1/2})$ let $\mathcal{U}_{(j,h)}$ be the index set of a [2] (mod 0)-partition $\{U_\ell\}_{\ell \in \mathcal{U}_{(j,h)}}$ of $T^n(I_j \cap O_h)$ into open sets such that

$$\operatorname{diam} U_\ell < \varepsilon_0, \forall \ell \in \mathcal{U}_{(j,h)}, \tag{5.4}$$

$V_* \subset U_\ell$ for some $\ell \in \mathcal{U}_{(j,h)}$ and such that, setting $V = T^n(I_j \cap O_h)$,

$$\frac{\sum_{\ell \in \mathcal{U}_{(j,h)}} \mathbf{m}(h(\partial_\varepsilon U_\ell \setminus \partial_\varepsilon V))}{\mathbf{m}(h(V))} \leq C_{\varepsilon_0}\varepsilon, \text{ for every } \varepsilon < \varepsilon_0. \tag{5.5}$$

For $(j, h) \in \mathcal{J} \times \mathcal{H}^n$ such that $\operatorname{diam} T^n(I_j \cap O_h) \leq \varepsilon_0$ set $\mathcal{U}_{(j,h)} = \varnothing$. Define

$$\mathcal{J}_n := \{(j, h, \ell) | (j, h) \in \mathcal{J} \times \mathcal{H}^n, \ell \in \mathcal{U}_{(j,h)}, \mathbf{m}(I_j \cap O_h) > 0\}.^3 \tag{5.6}$$

For every $j_n := (j, h, \ell) \in \mathcal{J}_n$, define $I_{j_n} := T^n(I_j \cap O_h) \cap U_\ell$ and $\rho_{j_n} : I_{j_n} \to \mathbb{R}^+$, $\rho_{j_n} := \rho_j \circ h \cdot Jh \cdot z_{j_n}^{-1}$, where $z_{j_n} := \int_{I_{j_n}} \rho_j \circ h Jh$. Define $\mathcal{T}^n\mathcal{G} := \{(I_{j_n}, \rho_{j_n})\}_{j_n \in \mathcal{J}_n}$ and associate to it the measure given by

$$w_{\mathcal{T}^n\mathcal{G}}(j_n) = z_{j_n} w_{\mathcal{G}}(j). \tag{5.7}$$

Remark 9 (Notation). To simplify notation throughout the rest of the paper we write w_{j_n} for $w_{\mathcal{T}^n\mathcal{G}}(j_n)$ and w_j for $w_{\mathcal{G}}(j)$.

Remark 10. If $\mathcal{G}$ is an (a_0, ε_0)–standard family, then so is $\mathcal{T}^n\mathcal{G}$ – a fact that is justified by Lemma 3.

Also, a simple change of variables shows that for every standard family $\mathcal{G}$ and every $n \in \mathbb{N}$, $|\mathcal{T}^n\mathcal{G}| = |\mathcal{G}|$. That is, the total weight does not change under iterations. We will make use of this fact throughout the article.

Lemma 1 (Artificial chopping avoiding a small set V_*)**.** *Suppose V is a bounded, open subset of $\mathbb{R}^d$, in the range of T, with* $\operatorname{diam} V > \varepsilon_0$*, $V_* \subset V$ is a subset of* $\operatorname{diam} V_* \leq \varepsilon_0/(4\sqrt{d})$ *and T satisfies **(1)** and **(2)**. Then, there exists a (mod 0)-partition $\{U_\ell\}_{\ell \in \mathcal{U}}$ of V into open sets such that* $\operatorname{diam} U_\ell \leq \varepsilon_0$ $\forall \ell \in \mathcal{U}$*, $V_* \subset U_\ell$ for some $\ell \in \mathcal{U}$, and*

$$\frac{\sum_{\ell \in \mathcal{U}} \mathbf{m}(h(\partial_\varepsilon U_\ell \setminus \partial_\varepsilon V))}{\mathbf{m}(h(V))} \leq C_{\varepsilon_0}\varepsilon, \text{ for every } \varepsilon < \varepsilon_0, \tag{5.8}$$

where $C_{\varepsilon_0} = e^{D \operatorname{diam}(X)^\alpha} 6d^{3/2} \cdot \varepsilon_0^{-1}$.

Proof. $\{U_\ell\}$ will be a family of sets formed by intersecting V with a grid of cubes of side-length $\varepsilon_0/(3\sqrt{d})$. Indeed, following [3, Proof of Theorem 2.1] and [2, p. 1349], let $\varepsilon_0' = \varepsilon_0/(3\sqrt{d})$ and given $0 \leq a_i < \varepsilon_0'$, $i = 1, \ldots, d$, consider the $(d-1)$–dimensional families of hyperplanes:

$$L_{a_i} = \{(x_1, \ldots, x_i, a_i + n_i\varepsilon_0', x_{i+1}, \ldots, x_{d-1}) | n_i \in \mathbb{Z}\}.$$

Denote the $(d-1)$–dimensional volume of $V \cap L_{a_i}$ by A_{a_i}. By Fubini theorem, $\int_0^{\varepsilon_0'} A_{a_i}\, da_i = \mathbf{m}(V)$. Therefore, $\exists a_i'$ such that $A_{a_i'} \leq \mathbf{m}(V)/\varepsilon_0'$. Let $L = \cup_i L_{a_i'}$ and denote the total $(d-1)$–dimensional volume of $L \cap V$ by A. Let $\mathcal{S} = \{S_\ell\}_{\ell \in \mathcal{U}}$ be the collection of cubes of the grid formed by L that intersect V. Let $U_\ell = S_\ell \cap V$, $\forall \ell \in \mathcal{U}$. Then we have

$$\mathbf{m}(\cup_{\ell \in \mathcal{U}}(\partial_\varepsilon U_\ell \setminus \partial_\varepsilon V)) \leq 2\varepsilon A \leq 2\varepsilon d\mathbf{m}(V)/\varepsilon_0' = 6d^{3/2}\varepsilon\mathbf{m}(V)/\varepsilon_0'.$$

[2]The existence of such a partition $\{U_\ell\}$ follows essentially from [3, Proof of Theorem 2.1] and [2, p. 1349], but for the sake of completeness it is also shown in Lemma 1. There may be many admissible choices for such "artificial chopping". One can make different choices at different iterations hence an n–th iterate of $\mathcal{G}$ is by no means uniquely defined (and this does not cause any problems).

[3]When $\mathcal{U}_{(j,h)} = \varnothing$, by (j, h, ℓ) we mean (j, h).

Now (5.8) follows by using the distortion bound (2.2). Finally, suppose $\operatorname{diam} V_* \le \varepsilon_0/(4\sqrt{d})$. Let $\ell' \in \mathcal{U}$ be such that $S_{\ell'} \cap V_* \ne \varnothing$. Consider the $2^d + 2d$ elements of $\mathcal{S}$ that share a face or a vertex with the cube $S_{\ell'}$. Denote them by $\{S_{\ell_j}\}_{j=1}^{2^d+2d+1}$ and let $S_{\ell_*} = \cup_{j=1}^{2^d+2d+1} S_{\ell_j}$, $U_{\ell_*} = \cup_{j=1}^{2^d+2d+1} U_{\ell_j}$. The set U_{ℓ_*} is contained in a cube of side-length $\varepsilon_0/\sqrt{d}$ hence it has diameter $\le \varepsilon_0$. Moreover, U_{ℓ_*} must contain the set V_* because otherwise V_* would contain one point from $S_{\ell'}$ and another point from $V \setminus S_{\ell_*}$. By construction the distance between these points would be greater than $\varepsilon_0/(3\sqrt{d})$, which contradicts $\operatorname{diam} V_* \le \varepsilon_0/(4\sqrt{d})$. Now, in the collection $\{U_\ell\}$, replace the elements $\{U_{\ell_j}\}_{j=1}^{2^d+2d+1}$ with the set U_{ℓ_*}. Then $V_* \subset U_{\ell_*}$, $\operatorname{diam} U_{\ell_*} \le \varepsilon_0$ and condition (5.8) is still satisfied. □

Next, we show the invariance of standard families under iteration, but first we state a simple lemma that provides a useful consequence of log-Hölder regularity (5.2).

Lemma 2 (Comparability Lemma)**.** *If $\rho : I \to \mathbb{R}^+$ satisfies $H(\rho) \le a$ for some $a \ge 0$ and $\operatorname{diam} I \le \varepsilon_0$, then for every $J, J' \subset I$ with $\mathbf{m}(J)\mathbf{m}(J') \ne 0$,*

$$\inf_I \rho \asymp_a \mathcal{A}_J \rho \asymp_a \mathcal{A}_{J'} \rho \asymp_a \sup_I \rho, \tag{5.9}$$

where $\mathcal{A}_J \rho = \mathbf{m}(J)^{-1} \int_J \rho$ is the average of ρ on J and $C_1 \asymp_a C_2$ means $e^{-a\varepsilon_0^\alpha} C_1 \le C_2 \le e^{a\varepsilon_0^\alpha} C_1$.

Proof. The lemma follows from the fact that if (I, ρ) satisfies $H(\rho) \le a$, then for every $x, y \in I$,

$$e^{-a\mathbf{d}(x,y)^\alpha} \rho(y) \le \rho(x) \le e^{a\mathbf{d}(x,y)^\alpha} \rho(y).$$

□

The following lemma together with Definition 5 justify the invariance of an (a_0, ε_0)–standard family under iteration.

Lemma 3. *Suppose (I, ρ) is an (a_0, ε_0)–standard pair and (I_n, ρ_n) is an image of it under T^n for some $n \in \mathbb{N}$, as in Definition 5. Then $\operatorname{diam}(I_n) \le \varepsilon_0$, $\int_{I_n} \rho_n = 1$ and*

$$H(\rho_n) \le a_0(\Lambda^{\alpha n} + a_0{}^{-1} D). \tag{5.10}$$

Proof. Using the definition of $H(\cdot)$, noting its properties under multiplication and composition, and using the expansion of the map, it follows that

$$H(\rho_{j_n}) \le H(Jh) + \Lambda^{\alpha n} H(\rho_j).$$

By (2.2) we have $H(Jh) \le D$, and by assumption $H(\rho_j) \le a_0$, finishing the proof of (5.10). □

The next lemma plays a crucial role in our arguments because it allows us to control the measure of points that map near the discontinuities.

Lemma 4 (Growth Lemma)**.** *Suppose $\varepsilon_0 > 0$, $n_0 \in \mathbb{N}$ and σ are as in our assumptions. Suppose $\mathcal{G}$ is an (a_0, ε_0)–standard family. Then for every $\varepsilon < \varepsilon_0$ we have*

$$|\partial_\varepsilon \mathcal{T}^{n_0} \mathcal{G}| \le (1 + e^{a_0 \varepsilon_0^\alpha} \sigma) |\partial_{\Lambda^{n_0} \varepsilon} \mathcal{G}| + \zeta_1 |\mathcal{G}| \varepsilon, \tag{5.11}$$

where $\zeta_1 = e^{a_0 \varepsilon_0^\alpha} C_{\varepsilon_0}$.

Proof. Suppose $\varepsilon < \varepsilon_0$. We write n for n_0. We have, by definition, $|\partial_\varepsilon \mathcal{T}^n \mathcal{G}| = \sum_{j_n} w_{j_n} \int_{\partial_\varepsilon I_{j_n}} \rho_{j_n}$. We split the sum into two parts according to whether $\mathcal{U}_{(j,h)} = \varnothing$

or $\mathcal{U}_{(j,h)} \neq \varnothing$. Suppose $\mathcal{U}_{(j,h)} = \varnothing$, that is $\operatorname{diam} T^n(I_j \cap O_h) \leq \varepsilon_0$ and $I_{j_n} = T^n(I_j \cap O_h)$. By a change of variables,

$$w_{j_n} \int_{\partial_\varepsilon I_{j_n}} \rho_{j_n} = w_j \int_{h(\partial_\varepsilon I_{j_n})} \rho_j.$$

For every $h \in \mathcal{H}^n$, since $h(\partial_\varepsilon I_{j_n}) \subset O_h$, we can write

$$h(\partial_\varepsilon I_{j_n}) \subset (h(\partial_\varepsilon I_{j_n}) \setminus \partial_{\Lambda^n\varepsilon} I_j) \cup (\partial_{\Lambda^n\varepsilon} I_j \cap O_h). \tag{5.12}$$

The integral over $\partial_{\Lambda^n\varepsilon} I_j \cap O_h$, and summed up over h and j is easily estimated by $|\partial_{\Lambda^n\varepsilon}\mathcal{G}|$. To estimate the integral of ρ_j over $h(\partial_\varepsilon I_{j_n}) \setminus \partial_{\Lambda^n\varepsilon} I_j$ we compare it, using Lemma 2, to $\int_{\partial_{\Lambda^n\varepsilon} I_j} \rho_j$ and we get

$$\int_{h(\partial_\varepsilon I_{j_n}) \setminus \partial_{\Lambda^n\varepsilon} I_j} \rho_j \leq e^{a_0\varepsilon_0^\alpha} \frac{\mathbf{m}(h(\partial_\varepsilon T^n(I_j \cap O_h)) \setminus \partial_{\Lambda^n\varepsilon} I_j)}{\mathbf{m}(\partial_{\Lambda^n\varepsilon} I_j)} \int_{\partial_{\Lambda^n\varepsilon} I_j} \rho_j$$

Note that if $\mathbf{m}(I_j \cap O_h) = 0$, then $\mathbf{m}(h(\partial_\varepsilon T^n(I_j \cap O_h))) = 0$ since $h(\partial_\varepsilon T^n(I_j \cap O_h)) \subset I_j \cap O_h$. By the controlled complexity condition (2.3),

$$\sum_{h \in \mathcal{H}^n} \frac{\mathbf{m}(h(\partial_\varepsilon T^n(I_j \cap O_h)) \setminus \partial_{\Lambda^n\varepsilon} I_j)}{\mathbf{m}(\partial_{\Lambda^n\varepsilon} I_j)} \leq \sigma. \tag{5.13}$$

Therefore,

$$\sum_{j \in \mathcal{J}} w_j \sum_{h \in \mathcal{H}^n} \int_{h(\partial_\varepsilon I_{j_n}) \setminus \partial_{\Lambda^n\varepsilon} I_j} \rho_j \leq e^{a_0\varepsilon_0^\alpha} \sigma |\partial_{\Lambda^n\varepsilon}\mathcal{G}|. \tag{5.14}$$

Now suppose that $\mathcal{U}_{(j,h)} \neq \varnothing$. By Definition 5, $\sum_{j_n} w_{j_n} \int_{\partial_\varepsilon I_{j_n}} \rho_{j_n}$ is bounded by $\leq \sum_j w_j \sum_{h,\ell} \int_{\partial_\varepsilon I_{j_n}} \rho_j \circ hJh$. Let us split the integral over two sets. Since $\partial_\varepsilon I_{j_n} \subset U_\ell$, we can write

$$\partial_\varepsilon I_{j_n} \subset (\partial_\varepsilon I_{j_n} \setminus \partial_\varepsilon T^n(I_j \cap O_h)) \cup (\partial_\varepsilon T^n(I_j \cap O_h) \cap U_\ell). \tag{5.15}$$

Consider the first term on the right-hand side of (5.15). We need to estimate the integral of $\rho_j \circ hJh$ on this set and sum over ℓ, h and j. Using a change of variables, the integral is

$$\int_{h(\partial_\varepsilon I_{j_n} \setminus \partial_\varepsilon T^n(I_j \cap O_h))} \rho_j.$$

Since $H(\rho_j) \leq a_0$, $h(\partial_\varepsilon I_{j_n} \setminus \partial_\varepsilon T^n(I_j \cap O_h)) \leq \operatorname{diam}(I_j) \leq \varepsilon_0$ and $\operatorname{diam}(h(T^n(I_j \cap O_h))) = \operatorname{diam}(I_j \cap O_h) \leq \operatorname{diam}(I_j) \leq \varepsilon_0$, we apply Lemma 2 to get

$$\int_{h(\partial_\varepsilon I_{j_n} \setminus \partial_\varepsilon T^n(I_j \cap O_h))} \rho_j \leq e^{a_0\varepsilon_0^\alpha} \frac{\mathbf{m}(h(\partial_\varepsilon I_{j_n} \setminus \partial_\varepsilon T^n(I_j \cap O_h)))}{\mathbf{m}(h(T^n(I_j \cap O_h)))} \int_{h(T^n(I_j \cap O_h))} \rho_j$$

Now we sum the above expression over ℓ, which is implicit in the notation $I_{j_n} = T^n(I_j \cap O_h) \cap U_\ell$. Using (5.5), we get

$$\leq e^{a_0\varepsilon_0^\alpha} C_{\varepsilon_0} \varepsilon \int_{I_j \cap O_h} \rho_j$$

Now we sum over h, multiply by w_j and sum over j. As a result we get the estimate $\leq e^{a_0\varepsilon_0^\alpha} C_{\varepsilon_0} \varepsilon |\mathcal{G}|$. Consider the second term on the right-hand side of (5.15). The contribution of this set is equal to $\sum_j w_j \sum_h \int_{h(\partial_\varepsilon T^n(I_j \cap O_h))} \rho_j$. But this was already included in the upper-bound estimate above (5.12)-(5.14), so we do not need to add it again. □

Recall from Remark 4 that n_0 is such that $\vartheta_1 := \Lambda^{n_0}(1 + e^{a_0\varepsilon_0^\alpha}\sigma) < 1$. Iterating Lemma 4 leads to the following. The proof is standard (uses (2.4)) so we omit it.

Corollary 2. *There exists $\zeta_2 \geq 0$ such that for every $k \in \mathbb{N}$ and $\varepsilon < \varepsilon_0$,*

$$|\partial_\varepsilon \mathcal{T}^{kn_0}\mathcal{G}| \leq (1 + e^{a_0\varepsilon_0^\alpha}\sigma)^k |\partial_{\Lambda^{kn_0}\varepsilon}\mathcal{G}| + \zeta_2|\mathcal{G}|\varepsilon. \tag{5.16}$$

Moreover, there exist $\zeta_2, \zeta_3 \geq 0$ such that for every $m \in \mathbb{N}$ that is not divisible by n_0 and for every $\varepsilon < \varepsilon_0$,

$$|\partial_\varepsilon \mathcal{T}^{m}\mathcal{G}| \leq \zeta_3(1 + e^{a_0\varepsilon_0^\alpha}\sigma)^{m/n_0} |\partial_{\Lambda^{m}\varepsilon}\mathcal{G}| + \zeta_4|\mathcal{G}|\varepsilon. \tag{5.17}$$

Proposition 1. *There exists $B_0 > 0$ such that for every $B \geq B_0$ there exists $n_{rec}(B) \in \mathbb{N}$ such that if $\mathcal{G}$ is an (a_0, ε_0, B)–proper standard family, then for every $m \geq n_{rec}(B)$, $\mathcal{T}^m\mathcal{G}$ is an $(a_0, \varepsilon_0, B_0)$–proper standard family.*

Proof. Choose $B_0 > \zeta_4$. Setting $\vartheta_2 = \vartheta_1^{1/n_0} < 1$ it follows from Corollary 2 that for every $m \in \mathbb{N}$ and $\varepsilon < \varepsilon_0$,

$$|\partial_\varepsilon \mathcal{T}^{m}\mathcal{G}| \leq |\mathcal{G}|\varepsilon(B\zeta_3\vartheta_2^m + \zeta_4). \tag{5.18}$$

Now choose $n_{rec}(B)$ so that $B\zeta_3\vartheta_2^{n_{rec}} + \zeta_4 \leq B_0$. □

Remark 11. $n_{rec} : [0, \infty) \to \mathbb{N}$ denotes the time it takes for an $(a_0, \varepsilon_0, \cdot)$–proper standard family to recover to an $(a_0, \varepsilon_0, B_0)$–proper standard family.

Definition 6. A set $I \subset X$ is said to be *δ–regular* if I is open and $\mathbf{m}(I \setminus \partial_\delta I) > 0$.

Lemma 5. *If I is a δ-regular set, then for every $x \in I \setminus \partial_\delta I$, the ball $\mathcal{B}(x, \delta)$ is contained in I.*

Proof. If I is δ-regular, then $I \setminus \partial_\delta I$ is non-empty. Consider a point $x \in I \setminus \partial_\delta I$ and the ball (in $\mathbb{R}^d$) $\mathcal{B}(x, \delta)$ of radius δ centred at x. If this ball is not entirely contained in I, then $\mathcal{B}(x, \delta) \cap I$ and $\mathcal{B}(x, \delta) \cap (\mathbb{R}^d \setminus \operatorname{cl} I)$ are non-empty open sets in $\mathbb{R}^d$. Furthermore since I is open and $\mathcal{B}(x, \delta)$ does not intersect ∂I, the union of the sets $\mathcal{B}(x, \delta) \cap (\mathbb{R}^d \setminus \operatorname{cl} I)$ and $\mathcal{B}(x, \delta) \cap I$ is $\mathcal{B}(x, \delta)$. This is a contradiction to $\mathcal{B}(x, \delta)$ being connected in $\mathbb{R}^d$. □

Remark 12. Define $\delta_0 = 1/(3B_0)$. It follows that if $\mathcal{G}$ is an $(a_0, \varepsilon_0, B_0)$–proper standard family, then more than (2/3) of its total weight is concentrated on δ_0-regular sets. That is,

$$\sum_{j \in \mathcal{J}_{reg}} w_j \geq \sum_j w_j \int_{I_j \setminus \partial_{\delta_0} I_j} \rho_j \geq 2/3, \tag{5.19}$$

where $\mathcal{J}_{reg}$ corresponds to indices j for which I_j is δ_0-regular.

6. Supplementary lemmas

This section contains supplementary lemmas for the proofs of our main theorems. The first lemma is taken from [2] and stated in a form that is suitable for our needs.

Lemma 6 (Sublemma C.1 of [2])**.** *Suppose I is a non-empty measurable bounded subset of $\mathbb{R}^d$ and E is a hyperplane cutting I into left and right parts I_l and I_r. Then $\forall \varepsilon \geq 0$ and $0 \leq \xi \leq 1$, we have*

$$\begin{aligned}\mathbf{m}(\{x \in I_l : \mathbf{d}(x, E) \leq \xi\varepsilon\} \setminus \{x \in I : \mathbf{d}(x, \partial I) \leq \varepsilon\}) \leq \\ \xi\mathbf{m}(\{x \in I_r : \mathbf{d}(x, \partial I) \leq \varepsilon\}).\end{aligned} \tag{6.1}$$

Let $V = \mathbf{m}(\mathcal{B}_1)$ denote the volume of the unit ball in $\mathbb{R}^d$ and set

$$c = \frac{\min\{1, V^{1/d}\}}{100\sqrt{d}}. \tag{6.2}$$

Lemma 7. *For every $\delta > 0$ and every open $Z \subset X$ that has a nice boundary and is contained in a d-dimensional cube of side-length $c\delta$, there exist a finite (mod 0)-partition $\mathcal{R} = \{R_j\}_{j=1}^N$ of X into open sets such that*

(1) $Z \in \mathcal{R}$.
(2) for every $1 \le j \le N$, $\sup_{\varepsilon>0} \varepsilon^{-1}\mathbf{m}(\partial_\varepsilon R_j) < \infty$,
(3) for every δ–regular set I, there exists $R \in \mathcal{R}$ s.t. $I \supset R$ and

$$\mathbf{m}(I \setminus R) \ \ge \ (1/2)\mathbf{m}(I); \tag{6.3}$$
$$\mathbf{m}(\partial_\varepsilon(I \setminus \operatorname{cl} R) \setminus \partial_\varepsilon I) \ \le \ 2d\mathbf{m}(\partial_\varepsilon I). \tag{6.4}$$

Proof. Let $\mathcal{S} = \{S_j\}$ denote a grid of open cubes in $\mathbb{R}^d$ with sides of length $c\delta$ parallel to the coordinate axes. Since X is bounded, the collection $\mathcal{R} = \{Z\} \cup \{S \cap (X \setminus Z) : S \in \mathcal{S}\}$ forms a finite (mod 0) partition of X into open sets. Now, if $R = Z$, then item (2) is satisfied because Z has nice boundary. If $R = S \cap (X \setminus Z)$, then

$$\begin{aligned}\mathbf{m}(\partial_\varepsilon R) &\le \mathbf{m}(\partial_\varepsilon S) + \mathbf{m}(\partial_\varepsilon(X \setminus Z)) \\ &\le \mathbf{m}(\partial_\varepsilon S) + \mathbf{m}(\partial_\varepsilon(X \setminus Z) \setminus \partial_\varepsilon X) + \mathbf{m}(\partial_\varepsilon X).\end{aligned}$$

By a simple calculation $\mathbf{m}(\partial_\varepsilon S) \le 2d\varepsilon(c\delta)^{d-1}$. Now item (2) follows because Z has a nice boundary.

Next, suppose $I \subset X$ is a δ-regular set. Then it contains a ball $\mathcal{B}$ of radius δ. Since, by construction, the diameter of Z and of elements of $\mathcal{S}$ are much smaller than δ, we can choose $R \in \mathcal{S}$ such that $R \subset \mathcal{B} \setminus Z$. Then (6.3) follows because

$$\mathbf{m}(R) \le (c\delta)^d = \frac{2^d c^d}{V}\mathbf{m}(\mathcal{B}(x, \frac{1}{2}\delta)) < \frac{1}{2}\mathbf{m}(\mathcal{B}(x, \frac{1}{2}\delta)) \le \frac{1}{2}\mathbf{m}(I).$$

Now since R is a cube in $\mathcal{S}$, each of its $2d$ sides can be continued as a hyperplane to cross I. By Lemma 6, the ε-boundary of each side contributes no more than the ε-boundary of I, verifying (6.4). □

Lemma 8 (Remainder family $\hat{\mathcal{G}}$)**.** *Suppose $\mathcal{G}$ is an $(a_0, \varepsilon_0, B_0)$–proper standard family. Let $\hat{\mathcal{G}}$ be the family obtained from $\mathcal{G}$ by replacing each (I, ρ) of weight w having a δ_0-regular domain containing an element $R = R(I) \in \mathcal{R}$ in its domain, as in item (3) of Lemma 7, with $\mathbf{m}(I \setminus R) \ne 0$, by $(I \setminus \operatorname{cl} R, \rho \mathbb{1}_{I\setminus \operatorname{cl} R}/\int_{I\setminus R}\rho)$ of weight $w\int_{I\setminus R}\rho$. Then $\hat{\mathcal{G}}$ is an $(a_0, \varepsilon_0, C_* B_0)$–proper standard family, where $C_* = 2(4de^{a_0\varepsilon_0^\alpha} + 1)e^{a_0\varepsilon_0^\alpha}$.*

Proof. This is a consequence of item (3) of Lemma 7. Indeed, assuming $\mathcal{G} = \{(I_j, \rho_j)\}$ with associated weights w_j, we have, $\forall \varepsilon < \varepsilon_0$,

$$\begin{aligned}|\partial_\varepsilon \hat{\mathcal{G}}| &\le \sum_j w_j \int_{\partial_\varepsilon(I_j \setminus \operatorname{cl} R)} \rho_j \le \sum_j w_j \left(\int_{\partial_\varepsilon(I_j\setminus \operatorname{cl} R)\setminus \partial_\varepsilon I_j} \rho_j + \int_{\partial_\varepsilon I_j} \rho_j \right) \\ &\le \sum_j w_j \left(e^{a_0\varepsilon_0^\alpha} \frac{\mathbf{m}(\partial_\varepsilon(I_j \setminus \operatorname{cl} R) \setminus \partial_\varepsilon I_j)}{\mathbf{m}(\partial_\varepsilon I_j)} \int_{\partial_\varepsilon I_j} \rho_j + \int_{\partial_\varepsilon I_j} \rho_j \right) \\ &\le (4de^{a_0\varepsilon_0^\alpha} + 1)|\partial_\varepsilon \mathcal{G}|,\end{aligned}$$

where in the second line we have used the Comparability Lemma 2 and in the last line we have used (6.4). Since $\mathcal{G}$ is B_0–proper, $|\partial_\varepsilon \mathcal{G}| \le B_0\varepsilon|\mathcal{G}|$; moreover (6.3) can be used to show that $|\mathcal{G}| \le 2e^{a_0\varepsilon_0^\alpha}|\hat{\mathcal{G}}|$. Indeed, by Lemma 2,

$$|\hat{\mathcal{G}}| \ge \sum_j w_j \int_{I_j\setminus R} \rho_j \ge \sum_j w_j e^{-a_0\varepsilon_0^\alpha} \frac{\mathbf{m}(I_j \setminus R)}{\mathbf{m}(I_j)} \int_{I_j} \rho_j \ge e^{-a_0\varepsilon_0^\alpha}(1/2)|\mathcal{G}|.$$

It follows that $|\partial_\varepsilon \hat{\mathcal{G}}| \le C_* B_0 \varepsilon |\hat{\mathcal{G}}|$. □

Lemma 9 (Remainder family $\hat{\mathcal{G}}$ in the presence of Z)**.** *Suppose $\mathcal{G}$ is an (a_0, ε_0, B)–proper standard family. Suppose $Z \subset Z' \subset X$, Z has a nice boundary and $\mathbf{m}(Z') > \mathbf{m}(Z)$. Let $\hat{\mathcal{G}}$ be the family obtained from $\mathcal{G}$ by replacing each (I, ρ) of weight w containing Z' in its domain, by $(I \setminus \mathrm{cl}\, Z, \rho \mathbb{1}_{I\setminus \mathrm{cl}\, Z} / \int_{I\setminus Z} \rho)$ of weight $w \int_{I \setminus Z} \rho$. Then $\hat{\mathcal{G}}$ is an (a_0, ε_0, B')–proper standard family, for some constant $B' > 0$.*

Proof. Assuming $\mathcal{G} = \{(I_j, \rho_j)\}$ with associated weights w_j, we have, $\forall \varepsilon < \varepsilon_0$,

$$\begin{aligned}
|\partial_\varepsilon \hat{\mathcal{G}}| &\le \sum_j w_j \int_{\partial_\varepsilon (I_j \setminus \mathrm{cl}\, Z)} \rho_j \le \sum_j w_j \left(\int_{\partial_\varepsilon (I_j \setminus \mathrm{cl}\, Z) \setminus \partial_\varepsilon I_j} \rho_j + \int_{\partial_\varepsilon I_j} \rho_j \right) \\
&\le \sum_j w_j \left(e^{a_0 \varepsilon_0^\alpha} \frac{\mathbf{m}(\partial_\varepsilon (I_j \setminus \mathrm{cl}\, Z) \setminus \partial_\varepsilon I_j)}{\mathbf{m}(\partial_\varepsilon I_j)} \int_{\partial_\varepsilon I_j} \rho_j + \int_{\partial_\varepsilon I_j} \rho_j \right) \\
&\le (e^{a_0 \varepsilon_0^\alpha} C_Z + 1) |\partial_\varepsilon \mathcal{G}|,
\end{aligned}$$

where in the second line we have used the Comparability Lemma 2. In the last line we have used a modified version of (6.4). Note that I_j is not necessarily δ_0-regular, but since $I_j \supset Z$ and Z has a nice boundary, $\mathbf{m}(\partial_\varepsilon (I \setminus \mathrm{cl}\, Z) \setminus \partial_\varepsilon I) \le C_Z \mathbf{m}(\partial_\varepsilon I)$.

Since $\mathcal{G}$ is B–proper, $|\partial_\varepsilon \mathcal{G}| \le B\varepsilon |\mathcal{G}|$. By Lemma 2, and since $\mathbf{m}(I_j \setminus Z) \ge \mathbf{m}(Z' \setminus Z) > 0$,

$$|\hat{\mathcal{G}}| \ge \sum_j w_j \int_{I_j \setminus Z} \rho_j \ge \sum_j w_j e^{-a_0 \varepsilon_0^\alpha} \frac{\mathbf{m}(I_j \setminus Z)}{\mathbf{m}(I_j)} \int_{I_j} \rho_j \ge \mathit{const.} |\mathcal{G}|.$$

It follows that $|\partial_\varepsilon \hat{\mathcal{G}}| \le B' \varepsilon |\hat{\mathcal{G}}|$ for some constant $B' > 0$. □

Lemma 10. *Let $\mathcal{R} = \{R_k\}_{k=1}^N$ be the partition from Lemma 7. There exists a constant $t > 0$ such that if $\mathcal{G} = \{(I_j, \rho_j)\}_{j \in \mathcal{J}}$ is an $(a_0, \varepsilon_0, B_0)$–proper standard family, then*

$$\sum_{j \in \mathcal{J}_{reg}} w_j \int_{R(I_j)} \rho_j \ge t \cdot \left(\sum_{j \notin \mathcal{J}_{reg}} w_j + \sum_{j \in \mathcal{J}_{reg}} w_j \int_{I_j \setminus R(I_j)} \rho_j \right), \tag{6.5}$$

where $\mathcal{J}_{reg}$ is the set of $j \in \mathcal{J}$ such that I_j is δ_0–regular

Proof. Since $\mathcal{G}$ is an $(a_0, \varepsilon_0, B_0)$–proper standard family, at least 2/3 of its weight is concentrated on (a_0, ε_0)–standard pairs (I, ρ), where I is a δ_0–regular set (recall that $\delta_0 = 1/(3B_0)$). By item (3) of Lemma 7, each such standard pair contains an element from the collection $\mathcal{R}$. Using this fact and the regularity of standard pairs (recall (5.9)), the left-hand side of (6.5) is

$$\ge (2/3) |\mathcal{G}| e^{-a_0 \varepsilon_0^\alpha} \mathbf{m}(R(I_j)) / \mathbf{m}(I_j) \ge (2/3) e^{-a_0 \varepsilon_0^\alpha} C_{ball}(\varepsilon_0)^{-1} \mathbf{m}(\mathcal{R}),$$

where $\mathbf{m}(\mathcal{R}) = \min_{1 \le k \le N} \mathbf{m}(R_k)$, and $C_{ball}(\varepsilon_0)$ denotes the measure of a ball of radius ε_0. Now consider the expression in the parentheses and on the right-hand side of (6.5). The first term of this expression is the total weight of the standard pairs that are not δ_0–regular so this term is $\le (1/3)|\mathcal{G}|$. The second term represents the weights of the remainders, after removing $\mathrm{cl}\, R(I_j)$, from each δ_0–regular I_j. This sum is

$$\le e^{a_0 \varepsilon_0^\alpha} \sum_j w_j \mathbf{m}(I_j \setminus R(I_j)) / \mathbf{m}(I_j) \le e^{a_0 \varepsilon_0^\alpha} |\mathcal{G}|.$$

So the expression in the parentheses and on the right-hand side of (6.5) is $\le |\mathcal{G}|(1/3 + e^{a_0 \varepsilon_0^\alpha})$. Therefore the inequality (6.5) is satisfied if we take:

$$t = \frac{(2/3)|\mathcal{G}| e^{-a_0 \varepsilon_0^\alpha} C_{ball}(\varepsilon_0)^{-1} \mathbf{m}(\mathcal{R})}{|\mathcal{G}|(1/3 + e^{a_0 \varepsilon_0^\alpha})} = \frac{(2/3)\mathbf{m}(\mathcal{R})}{C_{ball}(\varepsilon_0) e^{a_0 \varepsilon_0^\alpha} (1/3 + e^{a_0 \varepsilon_0^\alpha})}. \tag{6.6}$$

□

7. Proof of Theorem 1 and Corollary 1

Note that a_0, ε_0 and B_0, δ_0 are constants that depend on the map and are fixed once and for all once T is fixed. a_0, ε_0 are defined in (2.5) and B_0, δ_0 are defined in Section 5 in Proposition 1 and Remark 12.

Proof of Theorem 1. The following steps lead to our sought after inducing scheme.

(1) Fix $\delta = \delta_0$, $Z = \varnothing$ and consider the partition $\mathcal{R}$ of X given by Lemma 7. Let us focus on defining the inducing scheme on one element of this partition. The same can be done for all other partition elements and in a uniform way because $\mathcal{R}$ is finite. Fix $R \in \mathcal{R}$ and let $\mathcal{G}_0 = \{(R, \mathbb{1}_R/\mathbf{m}(R))\}$ and $w_0 = \mathbf{m}(R) > 0$. Due to item (2) of Lemma 7, the singleton family $\mathcal{G}_0$ with associated weight $\{w_0\}$ is an (a_0, ε_0, B)–proper standard family (Definition 4) for some constant $B > 0$ possibly larger than B_0.

(2) By Proposition 1, $\mathcal{G}_1 := \mathcal{T}^{n_{rec}(B)}\mathcal{G}_0$ is an $(a_0, \varepsilon_0, B_0)$–proper standard family.

(3) By item (3) of Lemma 7, every standard pair in $\mathcal{G}_1$ whose domain is δ_0-regular contains at least one element R' from the collection $\mathcal{R}$. "Stop" each δ_0-regular standard pair of $\mathcal{G}_1$ on its corresponding rectangle $R' \in \mathcal{R}$. By stopping we mean going back to R and defining the return time $\tau = n_{rec}(B)$ on the subset of R that maps onto R' under $\mathcal{T}^{n_{rec}(B)}$. By Lemma 10, applied to $\mathcal{G}_1$, the ratio of the removed weight from $\mathcal{G}_1$ to the weight of the remainder family (defined in Lemma 8), which we denote by $\hat{\mathcal{G}}_1$, is at least some positive constant t given by (6.6). Since the weight of standard pairs is preserved under iteration, this corresponds to defining τ on a subset $A \subset R$ such that $\mathbf{m}(A) \geq t \cdot \mathbf{m}(R \setminus A)$. Also, by Lemma 8, $\hat{\mathcal{G}}_1$ is an $(a_0, \varepsilon_0, C_* B_0)$–standard family.

(4) Just as in step (2), $\mathcal{G}_2 := \mathcal{T}^{n_{rec}(C_* B_0)}\hat{\mathcal{G}}_1$ is an $(a_0, \varepsilon_0, B_0)$–proper standard family so we can apply step (3) to it.

(5) Repeat the steps (3), (4) $\to$ (3), (4) $\to \cdots$, incrementing the indices accordingly during the process.

Applying the above inductive procedure, we will get a "stopping time" (or return time) $\tau : X \to \mathbb{N}$ defined on a (mod 0)-partition $\mathcal{P}'$ of X. $\mathcal{P}'$ is a refinement of the partition $\mathcal{P} \wedge \mathcal{R}$ and τ is constant on each element of $\mathcal{P}'$. The return time τ will have exponential tails because at each step (where the time between steps is universally bounded by $n_{rec}(B) + n_{rec}(C_* B_0)$) it is defined on a set $A \subset X$, where $\mathbf{m}(A) \geq t \cdot \mathbf{m}(X \setminus A)$. By construction the induced map has finitely many images which form a sub-collection of $\mathcal{R}$. Note that distortion bound is always maintained under iterations of T by assumptions **(1)** and **(2)** so we need not worry about it. □

Proof of Corollary 1. Let Z be one of the finitely many images of G that returns to itself infinitely many times. Suppose Z is minimal in the sense that no proper subsets of Z is an image of G. Let $\varsigma : Z \to \mathbb{N}$ be the first return time of G to Z and $\tilde{G} = G^{\varsigma} : Z \circlearrowleft$ be the associated first return map. Let $\mathcal{P}''$ be the partition of $\tilde{G}$, which is a refinement of $\mathcal{P}'$. Since G is Markov and Z is minimal, $\tilde{G}$ is full-branched. Define $\tilde{\tau} = \sum_{\ell=0}^{\varsigma-1} \tau \circ G^{\ell} : Z \to \mathbb{N}$, then $\tilde{G} = T^{\tilde{\tau}}$. Since G is a Markov map with finitely many images, ς has exponential tails and therefore $\tilde{\tau} : X \to \mathbb{N}$ also has exponential tails. □

8. Proof of Theorem 2

We start by a general fact which allows us to produce new recurrence times that are suitable for our needs.

Lemma 11. *Suppose Z is fully recurrent at times $\{\tilde{n}_j\}_{j=1}^K$ and $C_1, C_2 > 0$ are arbitrary constants. Then Z is fully recurrent at times $\{n_j\}_{j=1}^K$, where $n_1 \geq C_1$ and $n_{j+1} - n_j \geq C_2$ for every $j \in \{1, \dots, K-1\}$.*

Proof. Let $\{m_j\}_{j=1}^K \in \mathbb{N} \cup \{0\}$ be s.t. $m_1 \tilde{n}_K \geq C_1$ and $(m_{j+1} - m_j)\tilde{n}_1 \geq C_2$ and define

$$n_j := \tilde{n}_j + m_j \tilde{n}_K, \text{ if } 1 \leq j \leq K-1;$$

$$n_K := \tilde{n}_K + \sum_{j=1}^{K-1} n_j.$$

It follows from the properties of gcd that

$$\gcd(n_1, \dots, n_K) = \gcd(\tilde{n}_1, \dots, \tilde{n}_K).$$

Also, by definition, $n_1 \geq m_1 \tilde{n}_K \geq C_1$ and $n_{j+1} - n_j \geq (m_{j+1} - m_j)\tilde{n}_1 \geq C_2$.

Since Z covers itself when it returns at times $\{\tilde{n}_j\}$, the same holds at times $\{n_j\}$. It follows that Z is fully recurrent at times $\{n_j\}_{j=1}^K$. □

Proof of Theorem 2. We follow a line of reasoning similar to that of the proof of Theorem 1, but with some modifications when dealing with $R = Z$ mainly in order to achieve item (b) of Theorem 2.

(1) Fix $\delta = \delta_0$ and let c be as in (6.2). Let $Z = Z(c\delta)$ be as in the hypothesis of Theorem 2. Note that Z is also contained in a d-dimensional cube of side length $c\delta$. Let $\mathcal{R} = \mathcal{R}(\delta)$ be the partition given by Lemma 7. Let $\mathcal{G}_0 = \{(Z, \mathbb{1}_Z/\mathbf{m}(Z))\}$ and $w_0 = \mathbf{m}(Z)$. $\mathcal{G}_0$ is an (a_0, ε_0, B)–proper standard family for some $B > 0$. Applying Lemma 11 with $C_1 = n_{rec}(B)$ and $C_2 = n_{rec}(C_* B_0)$, it follows that Z is fully recurrent at times $\{n_j\}_{j=1}^K$, where

$$n_1 \geq n_{rec}(B) \text{ and } n_{j+1} - n_j \geq n_{rec}(C_* B_0),\ \forall j \in \{1, \dots, K-1\}.$$

(2) Let $\mathcal{G}_1 := \mathcal{T}^{n_1}\mathcal{G}_0$, taking $V_* = \operatorname{cl} Z$ as the set to avoid under $\mathcal{T}^{n_1}$ under artificial chopping. This can be done due to Lemma 1. Since $n_1 \geq n_{rec}(B)$, $\mathcal{G}_1$ is an $(a_0, \varepsilon_0, B_0)$–proper standard family.
   (2.1) There exists a standard pair in $\mathcal{G}_1$ whose domain contains Z. "Stop" it on Z. That is, define $\tau = n_1$ on
   $$A_1 := h_{n_1}(Z) \cap Z = \{x \in O_{h_{n_1}} \cap Z : T^{n_1}x \in Z\}.$$
   Note that $T^{n_1}A_1 = Z$. By Lemma 8, the remainder from $\mathcal{G}_1$, which we denote by $\hat{\mathcal{G}}_1$ is an $(a_0, \varepsilon_0, C_* B_0)$–proper standard family. Let $\mathcal{G}_2 = \mathcal{T}^{n_2 - n_1}\hat{\mathcal{G}}_1$. Since $n_2 - n_1 \geq n_{rec}(C_* B_0)$, $\mathcal{G}_2$ is an $(a_0, \varepsilon_0, B_0)$–proper standard family.
   (2.2) As before, define $\tau = n_2$ on $A_2 := h_{n_2}(Z) \cap Z$. Note that $T^{n_2}A_2 = Z$ and A_2 is disjoint from A_1 because $O_{h_{n_2}}$ is disjoint from $O_{h_{n_1}}$. By Lemma 8, the remainder from $\mathcal{G}_2$, which we denote by $\hat{\mathcal{G}}_2$ is an $(a_0, \varepsilon_0, C_* B_0)$–proper standard family. Let $\mathcal{G}_3 = \mathcal{T}^{n_3 - n_2}\hat{\mathcal{G}}_2$. Since $n_3 - n_2 \geq n_{rec}(C_* B_0)$, $\mathcal{G}_3$ is an $(a_0, \varepsilon_0, B_0)$–proper standard family.
   (2.3) We continue this process until we define $\tau = n_K$ on
   $$A_K := h_{n_K}(Z) \cap Z,$$
   which is disjoint from previous A_j's.
   Let $\hat{\mathcal{G}}_K$ be the remainder from $\mathcal{G}_K$. Note that $\hat{\mathcal{G}}_K$ is an $(a_0, \varepsilon_0, C_* B_0)$–proper standard family. Also note that $\forall j \in \{1, \dots, K\}$, $A_j \subset Z$ and $T^{n_j}A_j = Z$. Moreover, $\mathbf{m}(A_j) > 0$ because $\forall j \in \{1, \dots, K\}$ the inverse branches of T^{n_j} are non-singular, there are at most countably many such branches and $\mathbf{m}(Z) > 0$.

(3) We have achieved that

$$\gcd\left\{n:\mathbf{m}\left(\{\tau=n\}\cap\bigcup_{j=1}^{K}A_j\right)>0\right\}=1.$$

Also, by construction, T^τ maps each A_j onto Z in a one-to-one fashion.

We continue the construction of τ on the rest of Z, i.e. on $\hat{Z} := Z\setminus\bigcup_{j=1}^{K} A_j$, in such a way that it has exponential tails. We will do so by continuing to iterate $\hat{\mathcal{G}}_K$.

(4) Let $\mathcal{G}_{K+1} = \mathcal{T}^{n_{rec}(C_*B_0)}\hat{\mathcal{G}}_K$. Then $\mathcal{G}_{K+1}$ is an $(a_0, \varepsilon_0, B_0)$–proper standard family. By item (3) of Lemma 7, every standard pair in $\mathcal{G}_{K+1}$ whose domain is δ_0–regular contains an element R_k, $1 \le k \le N$, from the collection $\mathcal{R}$. "Stop" such standard pairs of $\mathcal{G}_{K+1}$ on $R_k \in \mathcal{R}$. By stopping we mean going back to $\hat{Z} \subset Z$ and defining the return time $\tau = n_K + n_{rec}(C_*B_0)$ on the subset of $\hat{Z}$ that maps onto R_k under $T^{n_K+n_{rec}(C_*B_0)}$. By Lemma 10, the ratio of the removed weight from $\mathcal{G}_{K+1}$ to the weight of the remainder family, which we denote by $\hat{\mathcal{G}}_{K+1}$, is at least some positive constant t given by (6.6). Note that since the total weight is preserved under iteration, this corresponds to defining τ on a subset $A \subset \hat{Z}$ such that $\mathbf{m}(A) \ge t \cdot \mathbf{m}(\hat{Z} \setminus A)$. Also, by Lemma 8, $\hat{\mathcal{G}}_{K+1}$ is an $(a_0, \varepsilon_0, C_*B_0)$–standard family.

(5) $\mathcal{G}_{K+2} := \mathcal{T}^{n_{rec}(C_*B_0)}\hat{\mathcal{G}}_{K+1}$ is an $(a_0, \varepsilon_0, B_0)$–proper standard family so we can apply step (4) to it.

(6) Repeat the steps (4), (5) $\to$ (4), (5) $\to \cdots$, incrementing the indices accordingly during the process. This procedure defines τ on $\hat{Z}$ up to a measure zero set of points (which includes points that map into ∂Z).

The above steps described how to define τ on Z. We have also explained how to define τ on the rest of the elements of $\mathcal{R}$ in the proof of Theorem 1. Putting these together we get the same statement as Theorem 1, but with the additional properties that $\gcd\{n : \mathbf{m}(\{\tau = n\} > 0\} = 1$, Z is one of the finitely many images of $G = T^\tau$ and that $G(Z) \supset Z$.

Let $\varsigma : Z \to \mathbb{N}$ be the first return time of G to Z and $\tilde{G} = G^\varsigma : Z \circlearrowleft$ be the associated first return map. Since $GA_j = T^\tau A_j = T^{n_j}A_j = Z$, $\forall j \in \{1,\dots,K\}$, it follows that $\varsigma = 1$ on the set $\bigcup_{j=1}^{K} A_j$.

Define $\tilde{\tau} = \sum_{\ell=0}^{\varsigma-1} \tau \circ G^\ell : Z \to \mathbb{N}$, then $\tilde{G} = T^{\tilde{\tau}}$. It follows from the previous paragraph that $\tilde{\tau} = \tau$ on $\bigcup_{j=1}^{K} A_j \subset Z$. This implies item (b). Item (a) and item (c) simply follow from the fact that G is a Markov map with finitely many states (hence ς has exponential tails) and $\tau : X \to \mathbb{N}$ has exponential tails. □

9. Proof of Theorem 3

The proof of Theorem 3 proceeds similarly to the proof of Theorem 2 except that in the initial step we need to define the stopping time pre-maturely because the initial family is not an $(a_0, \varepsilon_0, B_0)$–standard family and we cannot iterate to make it so. The remedy is to use Lemma 9 where we had previously used Lemma 8.

Proof. (1) Fix $\delta = \delta_0$ and let c be as in (6.2). Let $Z = Z(c\delta)$ be as in the hypothesis of Theorem 3. Note that Z is also contained in a d-dimensional cube of side length $c\delta$. Let $\mathcal{R} = \mathcal{R}(\delta)$ be the partition given by Lemma 7. Let $\mathcal{G}_0 = \{(Z, \mathbb{1}_Z/\mathbf{m}(Z))\}$ and $w_0 = \mathbf{m}(Z)$. $\mathcal{G}_0$ is an (a_0, ε_0, B)–proper standard family for some $B > 0$.

(2) Let $\mathcal{G}_1 := \mathcal{T}\mathcal{G}_0$, taking $V_* = \operatorname{cl} Z'$ as the set to avoid under $\mathcal{T}$ under artificial chopping. This can be done due to Lemma 1. Let h be as in the hypotheses of

Theorem 3. Define $\tau = 1$ on

$$A_1 := h(Z) \cap Z = \{x \in O_h \cap Z : Tx \in Z\}.$$

Note that $TA_1 = Z$. By Lemma 9, the remainder from $\mathcal{G}_1$, which we denote by $\hat{\mathcal{G}}_1$ is an (a_0, ε_0, B')–proper standard family for some $B' > 0$. Let $\hat{Z} := Z \setminus A_1$

(3) let $\mathcal{G}_2 := \mathcal{T}^{n_{rec}(B')}\mathcal{G}_0$. Then $\mathcal{G}_2$ is an $(a_0, \varepsilon_0, B_0)$–proper standard family.

(4) By item (3) of Lemma 7, every standard pair in $\mathcal{G}_2$ whose domain is δ_0-regular contains at least one element R' from the collection $\mathcal{R}$. "Stop" each δ_0-regular standard pair of $\mathcal{G}_2$ on its corresponding rectangle $R' \in \mathcal{R}$. By stopping we mean going back to Z and defining the return time $\tau = 1 + n_{rec}(B')$ on the subset of R that maps onto R' under $\mathcal{T}^{1+n_{rec}(B')}$. By Lemma 10, applied to $\mathcal{G}_2$, the ratio of the removed weight from $\mathcal{G}_2$ to the weight of the remainder family (defined in Lemma 8), which we denote by $\hat{\mathcal{G}}_2$, is at least some positive constant t given by (6.6). Since the weight of standard pairs is preserved under iteration, this corresponds to defining τ on a subset $A \subset R$ such that $\mathbf{m}(A) \geq t \cdot \mathbf{m}(R \setminus A)$. Also, by Lemma 8, $\hat{\mathcal{G}}_2$ is an $(a_0, \varepsilon_0, C_* B_0)$–standard family.

(5) Just as in step (3), $\mathcal{G}_3 := \mathcal{T}^{n_{rec}(C_* B_0)}\hat{\mathcal{G}}_2$ is an $(a_0, \varepsilon_0, B_0)$–proper standard family so we can apply step (4) to it.

(6) Repeat the steps (4), (5) $\to$ (4), (5) $\to \cdots$, incrementing the indices accordingly during the process.

The above steps described how to define τ on Z so that $\mathbf{m}(O_h \cap \{\tilde{\tau} = 1\}) > 0$. We have also explained how to define τ on the rest of the elements of $\mathcal{R}$ in the proof of Theorem 1. The rest of the proof is the same as the proof of Theorem 2. □

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