MIXING RATES FOR SYMPLECTIC ALMOST ANOSOV MAPS

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Abstract. We establish sharp bounds on the mixing rates of a class of two dimensional non-uniformly hyperbolic symplectic maps. This provides a primer on how to investigate such questions in a concrete example and, at the same time, it solves a controversy between previous rigorous results and numerical experiments.

1. Introduction

The study of decay of correlations for dynamical systems is a problem of paramount physical relevance. While the uniformly hyperbolic case is by now well understood and have been studied with very precise results, in the non-uniformly hyperbolic case there are still plenty of open problems. The basic idea to treat such cases is to induce. While many inducing schemes exist, the basic reference is the work of Lai Sang Young [20, 21]. This has allowed to obtain important and fairly general results, at least for rank one attractors, for maps with critical points (e.g see [18] and reference within). In this paper we consider non-uniformly hyperbolic maps without critical points. The one dimensional expanding map case, starting with [12] and continuing with [16, 17, 6], has witnessed many progressess that have developed into a rather satisfactory theory. Recently such a theory has been extended to important multidimensional expanding examples [8, 9, 5]. On the contrary the study of non-uniformly hyperbolic maps is still unsatisfactory. In particular, few example have been studied [7, 11, 2, 10] and only recently some general strategies are merging [13, 3].

It seems thus important to work out explicitly some relevant example to see how to develop the theory further. In particular, in [11] it was introduced a natural class of symplectic maps with a neutral fixed point. When such maps are smooth in [11] it was proven that smooth observables exhibit a decay of correlations with speed, at least, $n^{-2}(\log n)^4$. However numerical studies [1] suggest that the decay may be faster, leaving the doubt that the strategy used in [11] is largely suboptimal. Since the strategy is conceptually the same as used in [12], where it is optimal apart form a logarithmic factor, it is of clear interest to investigate if the suggestion coming from numerics is indeed correct, or it is a numerical artefact.

The present paper shows that the results in [11] are indeed sub-optimal and that the correlations decay with a faster power law. Moreover, we show that the power law that we obtain is optimal whereby putting to rest any previous doubt on the correct behaviour of the system.

The strategy used highlights the key ingredients that are necessary in order to achieve similar sharp results in different systems. In particular, it should be mentioned that the map considered is analytic and has a Markov partition and we take advantage of these facts in order to simplify the argument and present it in the

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simplest possible form. Yet, no real conceptual obstacle prevents one from trying to apply a similar strategy to a non-Markov or a piecewise $\mathcal{C}^{1+\alpha}$ map, where an array of different power-law decays should be present, see Remark 2.

The paper is organised as follows: in section 2 we present the class of maps we investigate, state our main result (Theorem 1) and we recall some relevant facts from the literature. In section 3 we describe various Markov partitions used in the following. Section 4 is devoted to the careful study of the return time when inducing on a set away from the fixed point. This is the core of the paper and, beside treating the current example, highlights the ingredients needed to obtain such results in more general models. At last, in section 5 we use the estimates of section 4, several facts from [11] and the general theory put forward in [3] to prove Theorem 1.

Remark 1 (Notation). Let $f, g : \mathbb{T}^2 \to \mathbb{R}$ be two functions. We write $f \ll g$ or $f = \mathcal{O}(g)$ if there exist constants $C, \delta > 0$, depending only on the map (2.1) and the Markov partition defined in Section 3, such that for every $x, y \in (0, \delta)$, $|f(x, y)| \leq C\,|g(x, y)|$. We write $f \asymp g$ if $f \ll g$ and $g \ll f$.

2. Preliminaries and results

We consider the class of maps $T : \mathbb{T}^2 \circlearrowleft$ from [11] and defined by

$$T(x, y) = (x + h(x) + y, h(x) + y), \tag{2.1}$$

where $h \in \mathcal{C}^\infty(\mathbb{T}^1, \mathbb{T}^1)$. We moreover require the following properties

1. $h(0) = 0$ (zero is a fixed point);
2. $h'(0) = 0$ (zero is a neutral fixed point)
3. $h'(x) > 0$ for each $x \neq 0$ (hyperbolicity)

Indeed condition (3) implies that, setting $K_0 = \{(v_1, v_2) \in \mathbb{R}^2 \;:\; v_1 v_2 \geq 0\}$, we have that, for all $(x, y) \in \mathbb{T}^2$, $D_{(x,y)}TK_0 \subset K_0$. Moreover, $D_{(x,y)}TK_0 \subset \mathrm{int}K_0 \cup \{0\}$ for all $x \neq 0$. Since $T(0, y) = (y, y)$, it follows that, if $(x, y) \neq 0$, then $D_{(x,y)}T^2K_0 \subset \mathrm{int}K_0 \cup \{0\}$. Hyperbolicity follows then by [19, Theorem 2.2].

Note that conditions (2–3) imply that zero is a minimum for h', which forces

$$h''(0) = 0; \quad h'''(0) \geq 0.$$

We will restrict to the generic case

4. $h'''(0) > 0$.

Hence, h, can be written, in a neighborhood of zero, as

$$h(x) = bx^3 + \mathcal{O}(x^5), \text{ for some } b > 0.$$

To simplify the following arguments we also assume

$$h(-x) = -h(x).$$

Note that this implies T is reversible (a physically meaningful property) with respect to the transformations

$$\Pi(x, -y - h(x)) := (x, -y - h(x)); \quad \Pi_1(x, y) := (-x, y + h(x)), \tag{2.2}$$

in the sense that $\Pi^2 = \Pi_1^2 = \mathbf{Id}$ and $\Pi T \Pi = \Pi_1 T \Pi_1 = T^{-1}$.

For the record,

$$T^{-1}(x, y) = (x - y, y - h(x - y)). \tag{2.3}$$

Remark 2. The choice $h \in \mathcal{C}^\infty$ is rather arbitrary. Most of hyperbolic theory applies to the case $h \in \mathcal{C}^{1+\alpha}$, $\alpha \in \mathbb{R}$, $\alpha > 0$. For example one could consider the cases in which, near zero, $h'(x) \sim |x|^\alpha$, possibly keeping the symmetry condition $h(-x) = -h(x)$. This would yield a large range of different behaviours of the decay of correlations, probably in analogy with what happens in the one dimensional case.

We do not pursue this venue here since it requires a considerable amount of extra work. In particular, one would have to extend all the relevant results obtained in [11] to the present case. However, what we do in the following constitutes a roadmap toward such an extension.

We are interested in studying the correlations between two observables:

$$\operatorname{cor}(\Phi, \Psi, n) = \int_{\mathbb{T}^2} \Phi \cdot \Psi \circ T^n \, d\mathbf{m} - \int_{\mathbb{T}^2} \Phi \, d\mathbf{m} \int_{\mathbb{T}^2} \Psi \, d\mathbf{m}.$$

Our main result consists in the following sharp estimate.

Theorem 1. *For all $\eta > 0$ sufficiently close to 1, there exist $C_1, C_2 > 0$ such that, for every $\Phi, \Psi \in \mathcal{C}^\eta(\mathbb{T}^2)$ satisfying $\int_{\mathbb{T}^2} \Phi \, d\mathbf{m} \int_{\mathbb{T}^2} \Psi \, d\mathbf{m} = 1$ (and supported away from $\mathbf{0}$ for the lower bound), the following estimate holds true. For every $n \geq 1$,*

$$C_1 \frac{\|\Phi\|_{\mathcal{C}^\eta} \|\Psi\|_{\mathcal{C}^\eta}}{(\log n) n^3} \leq |\operatorname{cor}(\Phi, \Psi, n)| \leq C_2 \frac{(\log n)^4 \|\Phi\|_{\mathcal{C}^\eta} \|\Psi\|_{\mathcal{C}^\eta}}{n^3}. \tag{2.4}$$

While, for every $\beta < 4$ the exists a constant $C_\beta > 0$ such that, for all $\Phi \in \mathcal{C}^\eta(\mathbb{T}^2)$ with $\int_{\mathbb{T}^2} \Phi \, d\mu = 0$ and $\Psi \in \mathcal{C}^\eta(\mathbb{T}^2)$, we have

$$|\operatorname{cor}(\Phi, \Psi, n)| \leq C_\beta n^{-\beta} \|\Phi\|_{\mathcal{C}^\eta} \|\Psi\|_{\mathcal{C}^\eta}.$$

Remark 3. The above result shows that, as suggested by some numerical experiments in [1], the estimate in [11] was off by one full power. Moreover, (2.4) shows that our estimate is essentially sharp.

To prove Theorem 1 it is necessary both a better understanding of the hyperbolic structure of the map and precise estimates on the behaviour of the map near its neutral fixed point.

The first is achieved by constructing a drastic refinement of the invariant cone field K_0: There exists two constants $K_+, K_- > 0$ such that the unstable direction $(1, u)$ at the point (x, y) satisfies

$$K_- \left(|x| + \sqrt{|y|}\right) \leq u \leq K_+ \left(|x| + \sqrt{|y|}\right). \tag{2.5}$$

Indeed, the unstable direction at $(x, y) = \xi$ must belong to the cone $D_{T^{-k}\xi} T^k K_0$, for each $k \in \mathbb{N}$. But, using (2.3), $D_{T^{-1}\xi} T^1 K_0$ is contained in the cone with boundary lines $(1 + h'(x - y), h'(x - y))$ and $(1, 1)$, so provided $|x - y| \geq \delta$, for some fixed δ, the claim is obvious for $K_+ < 1$ and K_- small enough. On the other hand, if $|x - y| \leq \delta$, then the lower boundary of $D_{T^{-2}\xi} T^2 K_0$ is above $(1 + h'(x - 2y + h(x - y)), h'(x - 2y + h(x - y)))$. Since $|x - 2y + h(x - y)| \geq |y| - (1 + \|h'\|_\infty)\delta$ we have again the claim provided $|y| \geq 2(1 + \|h'\|_\infty)\delta$ and K_- is small enough. It remains to verify the statement in a δ neighborhood of zero, which is done in [11, Proposition 4.1].

A similar statement holds for the stable direction.

As for the dynamics near zero, the first task is to understand the shape of the trajectories. This can be achieved with the introductions of an almost conserved quantity: a *quasi hamiltonian*.

For an initial point $(x, y) \in \mathbb{T}^2$ and $n \in \mathbb{N}$, denote $(x_n, y_n) = T^n(x, y)$. By (2.1),

$$x_{n+1} - x_n = y_n + h(x_n) = y_{n+1}. \tag{2.6}$$

We define the quasi-Hamiltonian as

$$H(x, y) = \frac{1}{2} y^2 - G(x) + \frac{1}{2} h(x) y - \frac{1}{12} h'(x) y^2 + \frac{1}{12} h(x)^2, \tag{2.7}$$

where $G(x) = \int_0^x h(z)\,dz$. Note that $G \geq 0$, $H(0,0) = 0$. A direct computation (if lazy see [11, Footnote 5]) yields, for every $(x,y) \in \mathbb{T}^2$,

$$|H(T(x,y)) - H(x,y)| \ll x^8 + y^4. \tag{2.8}$$

The dynamics along the trajctories is obviously dominated by dynamics on the stable and unstable manifolds of zero. The bounds on the cone filed (as well as (2.7), (2.8)) imply that they look like parabolas. The following Lemma corresponds to [11, Lemmata 3.1, 3.2].

Lemma 1 (Dynamics on the stable manifold)**.** *Denote* $A = (2/b)^{1/2}$. *Suppose* $x_0 \geq 0$ *is sufficiently small. Then, there exists a trajectory* $(x_n, y_n) = T^n(x_0, y_0)$, $n \in \mathbb{N} \cup \{0\}$, *that satisfies* $(x_n)_n \in \ell^2(\mathbb{N})$ *and*

$$\left| x_n - \frac{A}{n + A/x_0} \right| \leq \frac{B}{(n + A/x_0)^2}, \text{ for some } B \text{ and all } n \in \mathbb{N}; \tag{2.9}$$

moreover, y_0 *is a Lipschitz function of* x_0. *These trajectories form the local stable manifold of* $\mathbf{0} = (0,0)$.

3. Markov partitions

Using the symmetry (by which we mean reversiblility according to (2.2)), we can form a Markov partition for T consisting of three elements, as shown in Figure 1.

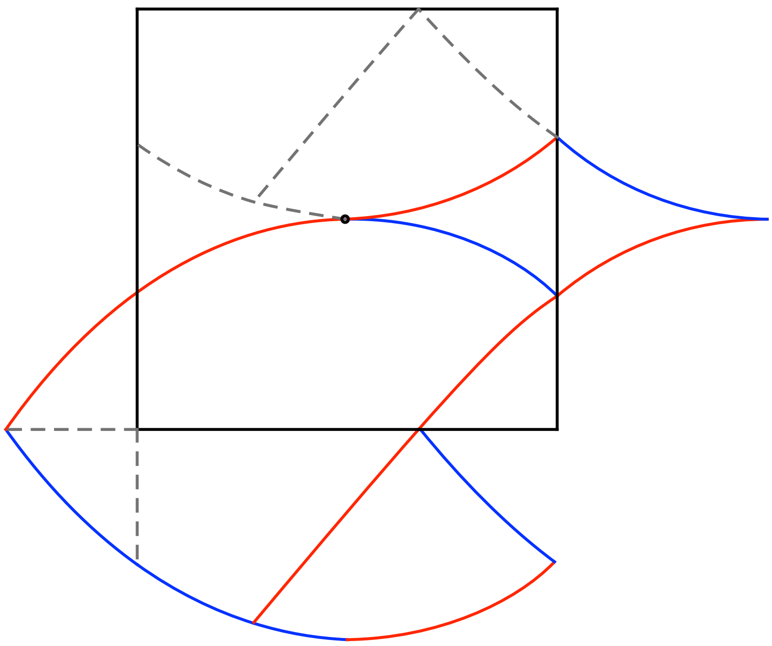

Figure 1. Fundamental domain of the Markov partition consisting of three rectangles. Note that the regions sticking out of the square can be $\mathbb{Z}^2$-translated back into the square to fill up the whole square. The blue (decreasing) and red (increasing) curves are pieces of the stable and unstable manifolds of $\mathbf{0}$, respectively.

We want to induce on a set away from zero; however, all the above partition elements touch zero. Hence, we refine the original partition forwards and backwards m times. We denote such a partition by $\mathcal{P}$. Let O denote the union of all the elements of $\mathcal{P}$ touching $\mathbf{0}$. By choosing m large enough we ensure that O is contained in a fixed but sufficiently small neighbourhood of $\mathbf{0}$.

It is easy to see that O is necessarily a union of four elements of $\mathcal{P}$ each of which have two sides consisting of local stable and unstable manifolds of $\mathbf{0}$. Since Π_1 maps the local unstable manifold of $\mathbf{0}$ to its local stable manifold, by continuity, if P_0 is one of the three elements of the original Markov partition, then $\Pi_1 P_0 = P_0$. Now, suppose $P \in \mathcal{P}$ is one of the four sets constituting O. It is of the form

$$P = P_0 \cap T^{-1}P_1 \cap \cdots \cap T^{-m}P_m \cap T\tilde{P}_1 \cap \cdots \cap T^m \tilde{P}_m,$$

where each P_j and $\tilde{P}_j$ is one of the three elements of the original Markov partition. Using the reversibility property of T with respect to Π_1, it follows that $\Pi_1 P$ is of the same form. Since $\Pi_1 P$ touches $\mathbf{0}$ and is a member of $\mathcal{P}$, it follows that it is one of the four elements constituting O. It follows that $\Pi_1 O = O$.

4. First return map

Denote $Y = \mathbb{T}^2 \setminus O$ and define $\varphi_{0,1} : Y \to \mathbb{N}$ to be the first return time to Y. The first return map $T_1 : Y \circlearrowleft$ is then defined by $T_1 = T^{\varphi_{0,1}}$. Note that Y is the union of elements of $\mathcal{P} \setminus \{O\}$ and if we let $R_N = \{\varphi_{0,1} = N\}$, then each R_N is a union of elements of $\mathcal{P}^N$. Let $\mathcal{P}_1$ denote the partition of Y whose elements are of the form $P \cap R_N$, where $P \in \mathcal{P} \setminus \{O\}$ and $N \in \mathbb{N}$. Notice that $\mathcal{P}_1$ is a (countably infinite) Markov partition for T_1.

Denote $Q = T^{-1}O \setminus O$. Then, $Q = \bigcup_{N \geq 2} R_N$. Our aim in this section is to estimate $\mathbf{m}(R_N)$ for large N (see Remark 6).

For large N, R_N consists of two parts one in the second quadrant ($x \leq 0, y \geq 0$) and the other in the fourth quadrant ($x \geq 0, y \leq 0$). Due to the symmetry we focus on the part of R_N contained in the second quadrant. We further consider two cases one corresponding to the part above the stable manifold of $\mathbf{0}$ (fat region); the other corresponding to the part below the stable manifold of $\mathbf{0}$ (thin region).

4.1. Analysis in the fat region.

First we analyze the dynamics in the fat region. In Section 4.2 we do a similar analysis for the thin region.

Fix $M > 0$ large (according to Lemma 3) and let

$$\mathbb{P}_M = \{(x, y) \in \mathbb{T}^2 : Mx^4 \leq H(x, y), x \leq 0, y \geq 0\}.$$

Due to the parabolic nature of stable and unstable manifolds [11, Lemma 3.5], in the fat region it holds $x^2 \ll y$. However, a better estimate holds for $(x, y) \in \mathbb{P}_M$ by taking M large.

Lemma 2. *For all $(x, y) \in \mathbb{P}_M$,*

$$x^2 \ll (1/M)y. \tag{4.1}$$

Proof. From (2.7), $H(x, y) \leq (1/2)y^2 + (1/2)h(x)^2 \ll y^2$. Also, by assumption, $Mx^4 \leq H(x, y)$. The result follows. □

Lemma 3. *For M sufficiently large and $(x, y) \in \mathbb{P}_M$,*

$$H(x, y) \asymp y^2. \tag{4.2}$$

Proof. By (2.7), $H(x, y) \geq (1/2)y^2 - G(x) + (1/2)h(x)y - (1/12)h'(x)y^2$. By (4.1) this expression is $\gg (1/2)M^2x^4 - x^4$. Choosing M sufficiently large implies the lower bound in (4.2). The upperbound was shown in the proof of Lemma 2. □

Lemma 4. *For $(x, y) \in \mathbb{P}_M$,*

$$\left| H^{1/2} \circ T(x, y) - H^{1/2}(x, y) \right| \ll y^3. \tag{4.3}$$

Proof. By (4.2) and (2.8) for $(x,y)\in\mathbb{P}_M$,

$$\begin{aligned}\left|H^{1/2}\circ T(x,y)-H^{1/2}(x,y)\right| &= \frac{|H\circ T(x,y)-H(x,y)|}{H^{1/2}\circ T(x,y)+H^{1/2}(x,y)}\\ &\ll \frac{|H\circ T(x,y)-H(x,y)|}{y}\\ &\ll \frac{x^8+y^4}{y}\ll y^3.\end{aligned}$$

□

Definition 1. Given an initial point $(x,y)\in Q$ let $E_k=E_k(x,y)=H(x_k,y_k)$,

$$n=\max\{k\in\mathbb{N}:x_k\le 0\},\quad \ell=\ell(x,y)=\min\{k\le n:(x_k,y_k)\in\mathbb{P}_M\}.$$

Remark 4. Note that in the estimates below both sides of $\ll$ are functions of the initial point (x,y) so the constants hidden in the notation $\ll$ do not depend on k.

Lemma 5. *For all $\ell\le k\le n$,*

$$\left|E_k^{1/2}-E_\ell^{1/2}\right|=\left|H^{1/2}(x_k,y_k)-H^{1/2}(x_\ell,y_\ell)\right|\ll|y_\ell|^2|x_\ell| \tag{4.4}$$

Proof. By (4.3),

$$\begin{aligned}\left|E_k^{1/2}-E_\ell^{1/2}\right| &= \left|H^{1/2}(x_k,y_k)-H^{1/2}(x_\ell,y_\ell)\right|\\ &\ll\sum_{j=\ell}^{k-1}|y_j|^3\ll|y_\ell|^2\sum_{j=\ell}^{k-1}(x_j-x_{j-1})\ll|y_\ell|^2|x_\ell|.\end{aligned} \tag{4.5}$$

□

Remark 5. If $(x_k,y_k)\in\mathbb{P}_M$, then by definition of $\mathbb{P}_M$ and (4.2),

$$|x_k|\le M^{-1/4}E_k^{1/4},\ \text{and}\ y_k\asymp E_k^{1/2}. \tag{4.6}$$

Lemma 6. *For $\ell\le k\le n$,*

$$y_k\asymp E_\ell^{1/2}. \tag{4.7}$$

Proof. Write $y_k=y_\ell-(y_\ell-y_k)$ and apply (4.4) and (4.6). □

Lemma 7.

$$E_{\ell-1}^{1/4}-E_\ell^{1/4}\ll E_\ell^{1/2}. \tag{4.8}$$

Proof. By definition of ℓ, $M^{-1/4}E_{\ell-1}^{1/4}\le -x_{\ell-1}$ and $M^{-1/4}E_\ell^{1/4}\ge -x_\ell$. Therefore, using (2.6) and (4.6),

$$M^{-1/4}E_{\ell-1}^{1/4}-M^{-1/4}E_\ell^{1/4}\le -x_{\ell-1}+x_\ell=y_\ell\ll E_\ell^{1/2}.$$

□

Now we relate the time and the energy.

Lemma 8. $n-\ell\asymp E_\ell^{-1/4}$.

Proof. By Remark 5 and Lemma 6,

$$(E_\ell/M)^{1/4}\ge -x_\ell\ge x_n-x_\ell=\sum_{k=\ell+1}^{n}y_k\gg(n-\ell-1)E_\ell^{1/2}$$

$$(E_{\ell-1}/M)^{1/4}\le -x_{\ell-1}\le x_{n+1}-x_{\ell-1}=\sum_{k=\ell}^{n+1}y_k\ll(n-\ell+1)E_\ell^{1/2}.$$

The result follows because, by (4.8),

$$E_{\ell-1}^{1/4} = E_\ell^{1/4} + (E_{\ell-1}^{1/4} - E_\ell^{1/4}) \geq E_\ell^{1/4} - \mathcal{O}(E_\ell^{1/2}) \gg E_\ell^{1/4}.$$

□

Lemma 9. $|x_\ell| \gg E_\ell^{1/4}$.

Proof. Using (2.6) and (4.7),

$$\begin{aligned} |x_\ell| &= |x_{\ell-1} + y_\ell| \geq |x_{\ell-1}| - |y_\ell| \\ &\geq (E_{\ell-1}/M)^{1/4} - \mathcal{O}(E_\ell^{1/2}) \gg E_{\ell-1}^{1/4} \gg E_\ell^{1/4}. \end{aligned}$$

□

Next, we relate ℓ to E_ℓ. Our strategy is to show that there exists a point (x^s, y^s) on the stable manifold of $\mathbf{0}$ whose trajectory "shadows" the trajectory of (x, y) for ℓ iterations; then, we use the dynamics on the stable manifold, Lemma 1, to estimate ℓ.

Denote a sufficiently long piece of the stable manifold of $\mathbf{0}$ by $\{(z, \gamma_s(z))\}$. Let $\{(z, U_\ell(z))\}$ be a sufficiently long piece of the unstable manifold of the point (x_ℓ, y_ℓ). Denote by $S_\ell = (\xi_\ell, \zeta_\ell)$ the point of intersection of U_ℓ and γ_s. That is, $U_\ell(\xi_\ell) = \gamma_s(\xi_\ell) = \zeta_\ell$.

Lemma 10. $0 \leq |\xi_\ell| - |x_\ell| \ll E_\ell^{1/4}$.

Proof. The lower bound clearly follows from (2.5) which implies that the unstable curves are increasing. To prove the upper bound, note that for every $\xi_\ell < z < x_\ell$,

$$U_\ell'(z) \geq K_-(|z| + \sqrt{|U_\ell(z)|}) \geq K_-(|x_\ell| + \sqrt{\zeta_\ell}).$$

From the geometry and using Lemma 9 and Lemma 6 we have

$$|\xi_\ell| - |x_\ell| \leq \frac{y_\ell - \zeta_\ell}{K_-(|x_\ell| + \sqrt{\zeta_\ell})} \ll \frac{y_\ell}{|x_\ell|} \ll \frac{E_\ell^{1/2}}{E_\ell^{1/4}} \ll E_\ell^{1/4}.$$

□

Remark 6. In the following lemma, in addition to previous restrictions on (x_0, y_0) being sufficiently close to $\mathbf{0}$, we need ℓ to be sufficiently large. This is accomplished by considering N sufficiently large beacuse this forces $(x_0, y_0) \in R_N$ to be sufficently close to the local stable manifold of $\mathbf{0}$, which in turn, by continuity, forces ℓ to be large.

Lemma 11. $\ell \asymp E_\ell^{-1/4}$.

Proof. Since $\{\xi_j\}_{j=0}^{\ell}$ lies on the stable manifold of $\mathbf{0}$, by (2.9), we have

$$\xi_\ell \asymp \frac{A}{\ell + A/\xi_0} \Rightarrow \ell \asymp \frac{A}{\xi_\ell} - \frac{A}{\xi_0}.$$

Since $|\xi_\ell| \ll E_\ell^{1/4}$, we have $\ell \gg E_\ell^{-1/4}$. On the other hand, by (4.8), we have

$$|\xi_{\ell-1}| \geq |x_{\ell-1}| \geq M^{-1/4} E_{\ell-1}^{1/4} \gg E_\ell^{1/4},$$

which shows that $\ell \ll E_\ell^{-1/4}$. □

For $k \in \mathbb{N}$, define $A_k = \{(x, y) \in Q : n(x, y) = k\}$. This is the set of points in $Q = T^{-1}O \setminus O$ that spend exactly k iterations on the left of the y-axis. We assume here that A_k is in the fat region and in the second quadrant.

We will need some extra information about the geometry of Q: By construction, and properties of the Markov partition, one unstable side of Q is the preimage of the other.

Lemma 12. *For sufficiently large k, A_k is the region bounded by the unstable sides of Q and the curves $T^{-k}(\{x=0\})$ and $T^{-(k+1)}(\{x=0\})$.*

Proof. For large k, the k-th preimage of the y-axis is a curve close to the stable manifold of $\mathbf{0}$. A_k is the region in Q that lies between the curves $T^{-k}(\{x=0\})$ and $T^{-(k+1)}(\{x=0\})$. □

Let us denote $C_k = T^k A_k$. For sufficiently large k, C_k is the region bounded by two unstable curves, the curve $T^{-1}(\{x=0\})$ and the y-axis.

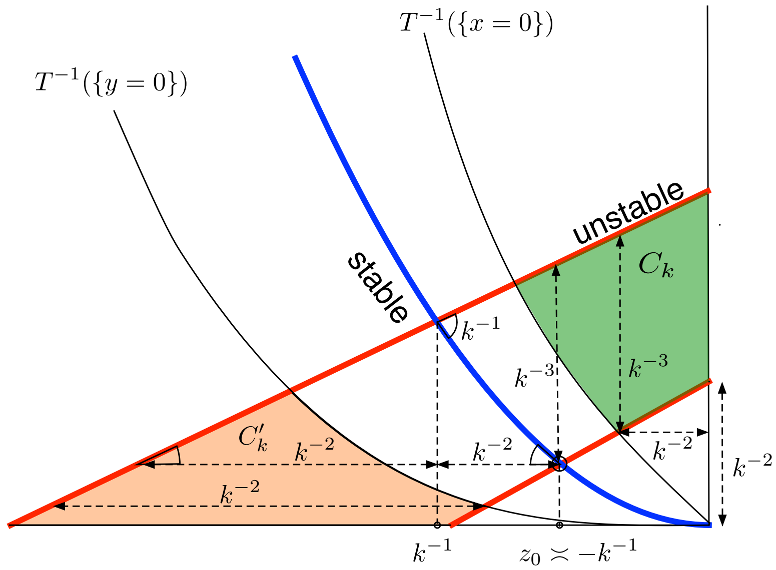

FIGURE 2. Proof of Lemma 13. Unstable curves are depicted as red increasing straight lines while stable curves (in blue) are decreasing, but in reality they are just smooth increasing and decreasing curves, respectively.

The following lemma gives an estimate on the vertical distance between the unstable sides of C_k; that is, on the length of a vertical line segment both of whose endpoints lie on the unstable sides of C_k.

Lemma 13. *The vertical distance between the unstable sides of C_k is $\asymp k^{-3}$.*

Proof. The upper and lower boundaries of C_k are formed by unstable manifolds that if continued to the left will intersect the immediate stable manifold of $\mathbf{0}$ at points whose x-coordinate is proportional to k^{-1} according to (2.9). Note that one unstable curve is the preimage of the other. At the points of intersection between the unstable manifolds and the stable manifold (shown in Figure 2), by (2.5), the angle between the manifolds is $\asymp k^{-1}$. Therefore the vertical distance between the two unstable curves is $\asymp k^{-3}$ at the intersection of the lower unstable curve and the stable curve (see Figure 2). Let us denote x-coordinate of this point by z_0. Let us label the two unstable curves by $(x, U_1(x))$ and $(x, U_2(x))$ and denote their slopes by $u_1(x)$ and $u_2(x)$. By the mean value inequality, it follows that

$$|u_2(x) - u_1(x)| \le |Du(\xi)| \, \|(x, U_2(x)) - (x, U_1(x))\| \,,$$

where $\xi \in \mathbb{T}^2$ is a point on the line segment connecting $(x, U_1(x))$ and $(x, U_2(x))$ and $(1, u(\xi))$ is the unstable vector at ξ.

By [11, Lemma 6.6 and Proposition 4.1], $|Du(\xi)| \ll \theta(\xi)^{-1}$, where $\theta(\xi) = u(\xi) + v(\xi)$ and $(1, -v(\xi))$ is the stable vector at ξ. By (2.5), $|Du(\xi)| \ll k$ in the region bounded by the stable curve, the two unstable curves and the y-axis.

Since $U_j'(x) = u_j(x)$, for $j = 1, 2$, we have shown that

$$|(U_2(x) - U_1(x))'| \ll k\, |U_2(x) - U_1(x)|\,.$$

Now, by Gronwall inequality,

$$U_2(z) - U_1(z) \le (U_2(z_0) - U_1(z_0))e^{Ck(z-z_0)}.$$

Recall that $-z_0 \asymp k^{-1}$, so for every $z \in (z_0, 0)$

$$U_2(z) - U_1(z) \le k^{-3} e^{Ck(z-z_0)} \ll k^{-3}. \tag{4.9}$$

Interchanging the role of z and z_0, we also get the lower bound,

$$U_2(z) - U_1(z) \ge k^{-3} e^{-Ck(z-z_0)} \ge k^{-3}e^{-C} \gg k^{-3}.$$

□

Lemma 14. *Suppose (x, y) is a point on the boundary of C_k which also lies on $T^{-1}\{x = 0\}$, then $|x| \asymp k^{-2}$.*

Proof. By Lemma 6, Lemma 8 and Lemma 11, $y \asymp k^{-2}$. Since $T^{-1}(\{x = 0\}) = \{(-y, y + h(y))\}$, it follows that $|x| \asymp k^{-2}$. □

Lemma 15. $\mathbf{m}(A_k) = \mathbf{m}(C_k) \asymp k^{-5}$.

Proof. This is a direct consequence of the previous two lemmas and the T-invariance of $\mathbf{m}$. □

Lemma 16. $|\pi_y(C_k)| \asymp k^{-3}$, *where π_y denotes projection onto the y-axis.*

Proof. We established in Lemma 13 that the vertical distance between the unstable sides of C_k is $\asymp k^{-3}$. It follows that $|\pi_y(C_k)| \gg k^{-3}$. To show the upper bound, it remains to take care of the inclination of the unstable sides. By (2.5), the angle of the unstable boundaries of C_k with the horizontal is $\asymp k^{-1}$ which means they can increase vertically by a factor of k^{-3} in a distance of k^{-2}. It follows that $|\pi_y(C_k)| \ll k^{-3}$. □

Remark 7. It follows from the previous lemmas that C_k is contained in a true rectangle (not a Markov rectangle) of vertical length $\asymp k^{-3}$ and of horizontal length $\asymp k^{-2}$.

Let $D_k := \Pi_1 C_k$. Due to the symmetry, $T^k D_k = T^k \Pi_1 \Pi_1 D_k = \Pi_1 T^{-k} \Pi_1 D_k = \Pi_1 T^{-k} C_k$. Therefore, D_k is the set of points on the right of the y-axis whose preimage is on the left of the y-axis and that spend exactly k iterations in O before mapping into $TO \setminus O$.

Lemma 17. *There exists $k_* \in \mathbb{N}$ such that for all sufficiently large k,*

$$TC_k \subset D_{k-k_*} \cup \cdots \cup D_{k+k_*}.$$

Proof. Using the definition of T, TC_k is vertically lower than C_k by an amount proportional to k^{-6} and the vertical distance between its unstable sides is $\asymp k^{-3}$. Also D_k has vertical height proportional to k^{-3} and has inclination (with respect to the horizontal) of k^{-3} (see proof of Lemma 16), as depicted in Figure 3. Since the proportionality constants are independent of k (i.e. they hold for all sufficiently large k), TC_k can be covered by finitely many D_k's. That is, there exists k_* such that $TC_k \subset D_{k-k_*} \cup \cdots \cup D_{k+k_*}$. □

Lemma 18. *For sufficiently large k,*

$$A_k \subset R_{2k-k_*} \cup \cdots \cup R_{2k+k_*}, \qquad R_{2k}, R_{2k+1} \subset A_{k-k_*} \cup \cdots \cup A_{k+k_*}.$$

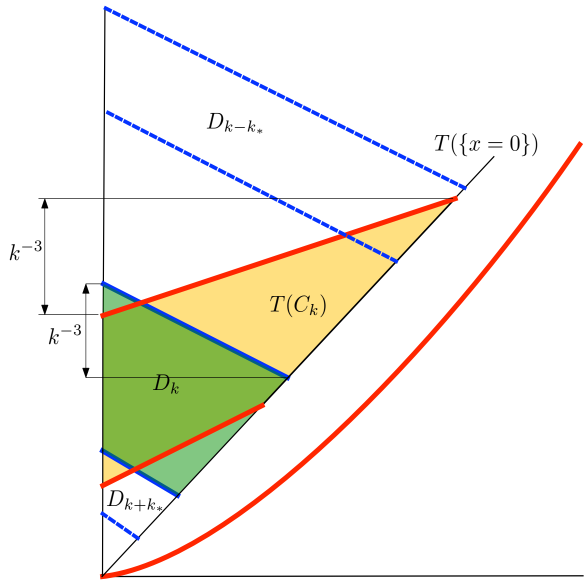

FIGURE 3. Proof of Lemma 17.

Proof. The first statement follows directly from the previous lemma. For the second statement, note that R_{2k+1} spends $2k$ iterations in O and this number of iterations is divided between left and right sides of the y-axis with the restriction that the number of iterations on the left and on the right can differ by at most k_* because of the previous lemma and the symmetry. This forces at most j iterations on the left and $2k-j$ iterations on the right, where $-k_* \le j \le k_*$. This implies that R_{2k+1} is a subset of $A_{k-k_*} \cup \cdots \cup A_{k+k_*}$. For the same reason R_{2k} must be a subset of $A_{k-k_*} \cup \cdots \cup A_{k+k_*}$. □

Proposition 1.

$$\mathbf{m}(R_N^{fat}) \asymp N^{-5}.$$

Proof. This is a direct consequence of the previous lemma and the estimate on the measure of A_k from Lemma 15. □

4.2. **Analysis in the thin region.** Analysis in the thin region is similar to the analysis we did in Section 4.1 for the fat region. Note that in this region the energy is negative. Fix M large (according to Lemma 20) and define

$$\mathbb{P}'_M = \{(x,y) \in \mathbb{T}^2 : My^2 \le |H(x,y)|, x \le 0, y \ge 0\}.$$

A priori, in the thin region, $y \ll x^2$ but in $\mathbb{P}'_M$ we have the following better estimate.

Lemma 19. *If $(x,y) \in \mathbb{P}'_M$ then*

$$y \ll (1/M)x^2. \tag{4.10}$$

Proof. By (2.7), $|H(x,y)| = -H(x,y) \le G(x) - (1/2)h(x)y + (1/12)h'(x)y^2 \ll x^4$. Also, by assumption, $My^2 \le |H(x,y)|$. Together, they imply the result. □

Lemma 20. *For M sufficiently large and $(x,y) \in \mathbb{P}'_M$,*

$$|H(x,y)| \asymp x^4 \tag{4.11}$$

Proof. By (2.7), $-H(x,y) \ge -(1/2)y^2 + \mathcal{O}(x^4)$. By (4.10), this expression is $\gg x^4$ if M is chosen sufficiently large. This proves the lower bound in (4.11). The upper bound was shown in the proof of Lemma 19. □

Lemma 21. *For* $(x,y) \in \mathbb{P}'_M$,

$$\left| |H \circ T(x,y)|^{1/2} - |H(x,y)|^{1/2} \right| \ll x^6. \tag{4.12}$$

Proof. By (4.11) and (2.8) for $(x,y) \in \mathbb{P}'_M$,

$$\begin{aligned}\left| |H \circ T(x,y)|^{1/2} - |H(x,y)|^{1/2} \right| &= \frac{|H \circ T(x,y) - H(x,y)|}{|H \circ T(x,y)|^{1/2} + |H(x,y)|^{1/2}} \\ &\ll \frac{x^8 + y^4}{x^2} \ll x^6.\end{aligned}$$

□

As before we define the following quantities but using $\mathbb{P}'_M$ instead of $\mathbb{P}_M$.

Definition 2. Given an initial point $(x,y) \in Q$ let $E_k = E_k(x,y) = H(x_k, y_k)$,

$$n = \max\{k \in \mathbb{N} : y_k \geq 0\}, \quad \ell = \ell(x,y) = \min\{k \leq n : (x_k, y_k) \in \mathbb{P}'_M\}.$$

Lemma 22. *For all* $\ell \leq k \leq n$,

$$\left| |E_k|^{1/2} - |E_\ell|^{1/2} \right| = \left| |H(x_k,y_k)|^{1/2} - |H(x_\ell,y_\ell)|^{1/2} \right| \ll |x_\ell|^3 y_\ell. \tag{4.13}$$

Proof. By (4.12),

$$\begin{aligned}\left| |E_k|^{1/2} - |E_\ell|^{1/2} \right| &= \left| |H(x_k,y_k)|^{1/2} - |H(x_\ell,y_\ell)|^{1/2} \right| \\ &\ll \sum_{j=\ell}^{k-1} x_j^6 \ll |x_\ell|^3 \sum_{j=\ell}^{k-1} (y_j - y_{j+1}) \ll |x_\ell|^3 y_\ell.\end{aligned}$$

□

Remark 8. If $(x_k, y_k) \in \mathbb{P}'_M$, then by definition of $\mathbb{P}'_M$ and (4.11),

$$|y_k| \leq M^{-1/2} |E_k|^{1/2}, \text{ and } x_k \asymp |E_k|^{1/4}. \tag{4.14}$$

Lemma 23. *For* $\ell \leq k \leq n$,

$$|x_k| \asymp |E_\ell|^{1/4} \tag{4.15}$$

Proof. Write $x_k = x_\ell + (x_k - x_\ell)$ and apply (4.13) and (4.14). □

Lemma 24.

$$|E_{\ell-1}|^{1/2} - |E_\ell|^{1/2} \ll |E_\ell|^{3/4}. \tag{4.16}$$

Proof. By definition of ℓ, $M^{-1/2} |E_{\ell-1}|^{1/2} \leq y_{\ell-1}$ and $M^{-1/2} |E_\ell|^{1/2} \geq y_\ell$. Therefore,

$$M^{-1/2} |E_{\ell-1}|^{1/2} - M^{-1/2} |E_\ell|^{1/2} \leq y_{\ell-1} - y_\ell \ll |x_{\ell-1}|^3 = |x_\ell - y_\ell|^3 \ll E_\ell^{3/4}.$$

□

Now we relate the time and the energy.

Lemma 25. $n - \ell \asymp |E_\ell|^{-1/2}$.

Proof. By Remark 8 and Lemma 23,

$$|E_\ell|^{1/2} \gg y_\ell \geq y_\ell - y_n \gg \sum_{k=\ell}^{n-1} |x_k|^3 \gg (n - \ell - 1) |E_\ell|^{3/4}$$

$$|E_{\ell-1}|^{1/2} \ll y_{\ell-1} \leq y_{\ell-1} - y_{n+1} \ll \sum_{k=\ell-1}^{n} |x_k|^3 = (n - \ell + 1) |E_\ell|^{3/4}.$$

The result follows because, by (4.16),

$$|E_{\ell-1}|^{1/2} = |E_\ell|^{1/2} + (|E_{\ell-1}|^{1/2} - |E_\ell|^{1/2}) \gg |E_\ell|^{1/2}.$$

□

We shall not use the following lemma, nevertheless we state it since it is a symmetrical statement to Lemma 9.

Lemma 26. $|y_\ell| \gg |E_\ell|^{1/2}$.

Proof. Using Remark 8 and (4.15),

$$\begin{aligned}|y_\ell| = |y_{\ell-1} + h(x_\ell)| &\gg |y_{\ell-1}| - |x_\ell|^3 \\ &\gg |E_{\ell-1}|^{1/2} - \mathcal{O}(|E_\ell|^{3/4}) \gg |E_\ell|^{1/2}.\end{aligned}$$

□

Let ξ_ℓ, ζ_ℓ be as in Lemma 10.

Lemma 27. $0 \le |x_\ell| - |\xi_\ell| \ll |E_\ell|^{1/4}$.

Proof. The lower bound clearly follows from (2.5) which implies that the unstable curves are increasing. To prove the upper bound, note that for every $\xi_\ell < z < x_\ell$,

$$U_\ell'(z) \ge K_-(|z| + \sqrt{|U_\ell(z)|}) \ge K_-(|x_\ell| + \sqrt{\zeta_\ell}).$$

Using Remark 8 and (4.15) we have

$$|x_\ell| - |\xi_\ell| \le \frac{y_\ell - \zeta_\ell}{K_-(|x_\ell| + \sqrt{\zeta_\ell})} \ll \frac{y_\ell}{|x_\ell|} \ll \frac{|E_\ell|^{1/2}}{|E_\ell|^{1/4}} \ll |E_\ell|^{1/4}.$$

□

Lemma 28. $\ell \asymp |E_\ell|^{-1/4}$.

Proof. Since $\{\xi_j\}_{j=0}^{\ell}$ lies on the stable manifold of $\mathbf{0}$, by (2.9), we have

$$\xi_\ell \asymp \frac{A}{\ell + A/\xi_0} \Rightarrow \ell \asymp \frac{A}{\xi_\ell} - \frac{A}{\xi_0}.$$

Since $|\xi_\ell| \le |x_\ell| \ll |E_\ell|^{1/4}$, we get $\ell \gg |E_\ell|^{-1/4}$. On the other hand, by (4.16), we have

$$|\xi_{\ell-1}| \ge |x_{\ell-1}| = |x_\ell - y_\ell| \gg |E_\ell|^{1/4} - |E_\ell|^{1/2} \gg |E_\ell|^{1/4},$$

which shows that $\ell \ll E_\ell^{-1/4}$. □

The above lemmas show that we can prove the analog of Proposition 1 in the thin region. The proof is very similar, nevertheless we take the reader through it.

For $k \in \mathbb{N}$, define $A_k' = \{(x, y) \in Q : n(x, y) = k\}$. This is the set of points in $Q = T^{-1}O \setminus O$ that spend exactly k iterations on the top of the x-axis. We assume here that A_k is in the thin region and in the second quadrant.

Lemma 29. *For sufficiently large k, A_k' is the region bounded by the unstable sides of Q and the curves $T^{-k}(\{y = 0\})$ and $T^{-(k+1)}(\{y = 0\})$.*

Proof. For large k, the k-th preimage of the x-axis is a curve close to the stable manifold of $\mathbf{0}$. A_k' is the region in Q that lies between the curves $T^{-k}(\{y = 0\})$ and $T^{-(k+1)}(\{y = 0\})$. □

Let us denote $C_k' = T^k A_k'$. For sufficiently large k, C_k' is the region bounded by two unstable curves, the curve $T^{-1}(\{y = 0\})$ and the x-axis. See Figure 2.

The following lemma gives an estimate on the horizontal distance between the unstable sides of C_k'; that is, on the length of a horizontal line segment both of whose endpoints lie on the unstable sides of C_k'.

Lemma 30. *The horizontal distance between the unstable sides of C'_k is $\asymp k^{-2}$.*

Proof. The upper and lower boundaries of C'_k are formed by unstable manifolds that if continued to the right will intersect the immediate stable manifold of $\mathbf{0}$ at points whose x-coordinate is proportional to k^{-1} according to (2.9) (and using (2.5)). Note that one unstable curve is the preimage of the other. Consider the point of intersection of the bottom unstable manifold and the stable manifold shown in Figure 2. The horizontal line segment between this point and the the top unstable manifold consists of two subsegments both of which have length $\asymp k^{-2}$. In fact, the right subsegment has length $\asymp k^{-2}$ due to (2.9). The left subsegment has also length $\asymp k^{-2}$ because the angles shown in Figure 2 at the endpoints of the horizontal segment are both $\asymp k^{-1}$. So the horizontal distance between the two unstable curves at this level is $\asymp k^{-2}$.

Now a similar argument to the proof of Lemma 13 (interchanging the role of x and y axes) implies that the horizontal distance between the unstable curves remains $\asymp k^{-2}$ all the way down to the x-axis. □

Lemma 31. *Suppose (x, y) is a point on the boundary of C'_k which also lies on $T^{-1}\{y = 0\}$, then $y \asymp k^{-3}$.*

Proof. By Lemma 23, Lemma 28 and Lemma 25, $-x \asymp k^{-1}$. Since $T^{-1}(\{y = 0\}) = \{(x, -h(x))\}$, it follows that $y \asymp k^{-3}$. □

Lemma 32. $\mathbf{m}(A'_k) = \mathbf{m}(C'_k) \asymp k^{-5}$.

Proof. This is a direct consequence of the previous two lemmas and the T-invariance of $\mathbf{m}$. □

Lemma 33. $|\pi_x(C'_k)| \asymp k^{-2}$, *where π_x denotes projection onto the x-axis.*

Proof. We established in Lemma 30 that the horizontal distance between the unstable sides of C'_k is $\asymp k^{-2}$. It follows that $|\pi_x(C'_k)| \gg k^{-2}$. To show the upper bound, it remains to take care of the inclination of the unstable sides. By (2.5), the angle of the unstable boundaries of C_k with the horizontal is $\asymp k^{-1}$ which means they can increase horizontally by a factor of k^{-2} in a vertical distance of k^{-3}. It follows that $|\pi_x(C_k)| \ll k^{-2}$. □

Remark 9. It follows from the previous lemmas that C'_k is contained in a true rectangle (not a Markov rectangle) of vertical length $\asymp k^{-3}$ and of horizontal length $\asymp k^{-2}$.

Let $D'_k := \Pi_1 C'_k$. Due to the symmetry, D'_k is the set of points on the bottom of the x-axis whose preimage is on the top of the x-axis and that spend exactly k iterations in O before mapping into $TO \setminus O$.

Lemma 34. *There exists $k_* \in \mathbb{N}$ such that for all sufficiently large k,*

$$TC'_k \subset D'_{k-k_*} \cup \cdots \cup D'_{k+k_*}.$$

Proof. Using the definition of T, TC'_k is shifted to the left horizontally by an amount proportional to k^{-6} with respect to C'_k and the horizontal distance between its unstable sides is $\asymp k^{-2}$. Also D'_k has horizontal length proportional to k^{-2} and has inclination (with respect to the vertical) of k^{-2} (see proof of Lemma 33) as depicted in Figure 4. Since the proportionality constants are independent of k, TC'_k can be covered by finitely many D'_ks. That is, there exists k_* such that $TC'_k \subset D'_{k-k_*} \cup \cdots \cup D'_{k+k_*}$. □

Lemma 35. *For sufficiently large k,*

$$A'_k \subset R_{2k-k_*} \cup \cdots \cup R_{2k+k_*}, \qquad R_{2k}, R_{2k+1} \subset A'_{k-k_*} \cup \cdots \cup A'_{k+k_*}.$$

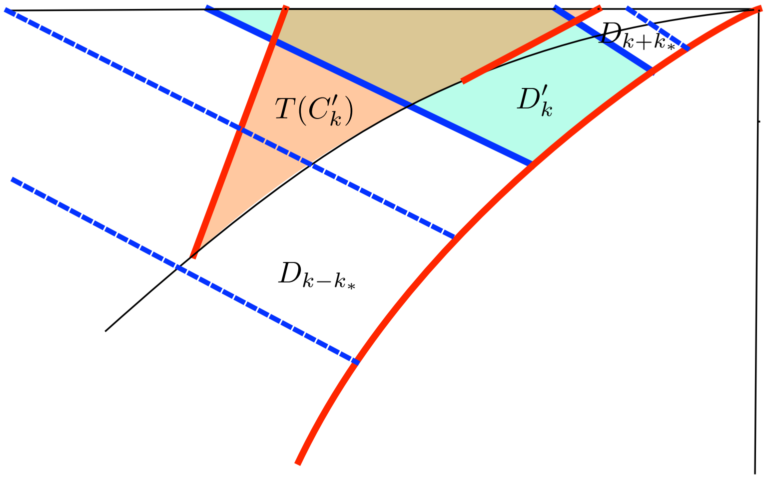

FIGURE 4. Proof of Lemma 34.

Proof. The first statement follows directly from the previous lemma. For the second statement, note that R_{2k+1} spends $2k$ iterations in O and this number of iterations is divided between top and bottom sides of the x-axis with the restriction that the number of iterations on the top and on the bottom can differ by at most k_* because of the previous lemma and the symmetry. This forces at most j iterations on the top and $2k-j$ iterations on the bottom, where $-k_* \le j \le k_*$. This implies that R_{2k+1} is a subset of $A'_{k-k_*} \cup \cdots \cup A'_{k+k_*}$. For the same reason R_{2k} must be a subset of $A'_{k-k_*} \cup \cdots \cup A'_{k+k_*}$. □

Proposition 2.

$$\mathbf{m}(R_N^{thin}) \asymp N^{-5}.$$

Proof. This is a direct consequence of the previous lemma and the estimate on the measure of A'_k from Lemma 32. □

Adding the two estimates of Proposition 1, Proposition 2 and symmetrical estimates for other regions, we get:

Proposition 3.

$$\mathbf{m}(\{\varphi_{0,1} = N\} \cap Q) \asymp N^{-5}.$$

5. MIXING RATES

To obtain upper and lower bounds on mixing rates we further induce T_1 (the first return map of T to Y) to a two-sided Young tower. This will allows us to apply [3, Theorem 7.4].

In order to induce T_1 to a two-sided Young tower with exponential tails, we check conditions (P1)–(P5) of [20, Section 1] for T_1. We will see that hese conditions follow from the existence of a finite Markov partition for T and the (non-uniform) hyperbolicity estimates for T established in [11].

Recall from Section 4 that T_1 has a countable Markov partition $\mathcal{P}_1$, which is a certain refinement of the finite parition $\mathcal{P} \setminus \{O\}$. We take $Q = \bigcup_{N\ge 2} R_N$ as the set with hyperbolic product structure required by [20, (P1)]. We claim that each of the sets R_N return under iterations of T_1 and u-cross the set Q. Furthermore, there is a fixed time N_* before which this happens. This claim follows from the construction of R_N and the existence of the finite Markov partition for T. Indeed the image of each R_N under T_1 must u-cross elements of $\mathcal{P}$. By finiteness of $\mathcal{P}$, the

Markov property and the ergodicity of T_1, in at most finitely many more iterations of T_1, R_N must also u-cross Q in which case we stop and define the return time accordingly. The leftover is again a union of sets that u-cross elements of $\mathcal{P}$, so we can repeat the same argumet for them. Since the time between the stopping times is uniformly bounded an due to boundedness of distortion, the tail of the stopping times will be exponentially small in the number of iterations. We have established [20, (P2)].

It remains to establish [20, (P3)-(P5)].

Hyperbolicity: [20, (P3) and (P4)(a)] require exponential contraction of T_1 along stable manifolds and backward contraction along unstable manifolds. Both statments follow from the estimates on expansion and contraction rates of the original map T in the neighbourhood O of $\mathbf{0}$. These estimates are obtained in [11, Lemma 5.1, 5.2 and Corollary 5.3]. Essentially, a vector in the unstable cone at $\xi \in R_N$ expands by a factor proportional to N^2 under $T_1|_{R_N} = T^N$. This implies uniform expansion on $\cup_{N=N_0}^{\infty} R_N$ for some N_0. We have also uniform expansion on the other finitely many R_N's by the uniform hyperbolicity of the original T away from $\mathbf{0}$. Similar estimate holds for backward expansion along the stable direction [11, Corollary 5.3]. Since T preserves Lebesgue measure, similar estimates hold for contraction along stable and backward expansion along unstable manifolds (see [11, Lemma 5.2]).

Distortion bounds: Let $J^uT(\xi) = \|D_\xi T(1,u)\| \,/\, \|(1,u)\|$ denote the factor of expansion on the unstable manifold of ξ. By direction calculation,

$$J^uT(\xi) = \left(\frac{(1 + h'(x) + u(\xi))^2 + (h'(x) + u(\xi))^2}{1 + u^2(\xi)} \right)^{1/2}.$$

Since h is smooth and u is differentiable on $\mathbb{T}^2 \setminus \{\mathbf{0}\}$, with a uniform bound on the derivative, it follows that there exists $C > 0$ such that $|\nabla \log J^uT(\xi)| \le C$. Now, by the mean value inequality, it follows that if η is another point on the unstable manifold of ξ, then

$$\frac{J^uT(\xi)}{J^uT(\eta)} \le e^{C\mathbf{d}(\xi,\eta)}.$$

The distortion property for T_1 now follows if we can ensure the same inequality along the orbit of ξ, η as they move through the region O, but this follows by chain rule and the estimate on the expansion factor along the orbit, which is $\ge CN^2$ (for some constant $C > 0$) where the points spend N iterations in O. Therefore the distortion property also holds for T_1. Conditions (P4)(b) and (P5)(a) follow from this distortion property and expansion and contraction along stable and unstable directions. Note that the estimates on the slope of stable and unstable manifolds (2.5) allow one to relate the distance $\mathbf{d}(\cdot,\cdot)$ to the distance along stable or unstable directions.

Regularity of the stable holonomy: Property (P5)(b) of [20] requires the existence and absolute continuity of the stable holonomy as well as an asymptotic formula for the Jacobian of the holonomy. All of these properties follow from the uniform hyperbolicity and distrotion estimates for T_1 established above. We refer the reader to [15, Proof of Theorem 3.1].

We have finished checking conditions (P1)-(P5) of [20] from which follows the existence of a two sided Young tower with exponential tails (for T_1). Let $\varphi_{1,2}$ be the stopping time defined above and $T_2 = T_1{}^{\varphi_{1,2}}$. Let us denote the full return time by $\varphi_{0,2} : Q \to \mathbb{N}$, $\varphi_{0,2}(x) = \sum_{k=0}^{\varphi_{1,2}(x)-1} \varphi_{0,1}(T_1^k(x))$. That is $T_2 = T^{\varphi_{0,2}}$. From standard computations $\varphi_{0,2}$ will have the same tail of return times as $\varphi_{0,1}$ with an additional $[\log(n)]^\alpha$ (for some $\alpha > 1$) factor. Note that since T is mixing, it follows that $\gcd(\varphi_{0,2}) = 1$.

The article [3] requires one more condition. We need to check that there exist $C > 0$ and $\theta \in (0,1)$ such that for every $z, z' \in Q$ and $n \geq 1$,

$$\mathbf{d}({T_2}^n z, {T_2}^n z') \leq C(\theta^n + \theta^{s(z,z')-n}). \tag{5.1}$$

This follows from uniform hyperbolicity. We can choose $C = \operatorname{diam}(Q)$ and $\theta = \max(\lambda_u^{-1}, \lambda_s)$, where λ_u and λ_s are respectively the worst expansion and contraction of the map T_2. Then (5.1) follows from T_2 being a uniformly hyperbolic map.

Now we are in a position to apply [3, Theorem 7.4].

Proof of Theorem 1. Denote $\bar{\varphi}_{0,2} = \int_Q \varphi_{0,2}\, d\mathbf{m}$. By [3, Theorem 7.4, Proposition 7.3], for all $\eta > 0$ sufficiently close to 1 there exists a constant $C > 0$ such that for every $\Phi, \Psi \in \mathcal{C}^\eta(\mathbb{T}^2)$ satisfying $\int_{\mathbb{T}^2} \Phi\, d\mathbf{m} \int_{\mathbb{T}^2} \Psi\, d\mathbf{m} = 1$ (and supported away from $\mathbf{0}$ for the lower bound)

$$\left| \operatorname{cor}(\Phi, \Psi, n) - \bar{\varphi}_{0,2} \sum_{j>n} \mathbf{m}(\varphi_{0,2} > j) \right| \leq C \left\|\Phi\right\|_{\mathcal{C}^\eta} \left\|\Psi\right\|_{\mathcal{C}^\eta} (\gamma_n + \zeta_{\beta'}(n)), \tag{5.2}$$

where

- β' can be taken any number $< \beta$ and β is the polynomical rate of decay of $\mathbf{m}(\varphi_{0,1} > n)$, which for us, by Proposition 3, $\beta = 4$.
- $\gamma_n \leq C n^{-\beta'} \log n$, by [3, Proposition 3.2].
- $\zeta_{\beta'}(n) = n^{-\beta'}$, for any $\beta' > 2$ by [3, equation (1.3)].

Also, by [3, Proposition 5.1] and Proposition 3,

$$C_1 (\log j)^{-1} j^{-4} \leq \mathbf{m}(\varphi_{0,2} > j) \leq C_2 (\log j)^4 j^{-4}.$$

Therefore, absorbing $\bar{\varphi}_{0,2}$ into the constants C_1, C_2 (whose values from one occurence to the other are not necessarily the same), (5.2) implies

$$C_1 \left\|\Phi\right\|_{\mathcal{C}^\eta} \left\|\Psi\right\|_{\mathcal{C}^\eta} n^{-3} (\log n)^{-1} \leq |\operatorname{cor}(\Phi, \Psi, n)| \leq C_2 \left\|\Phi\right\|_{\mathcal{C}^\eta} \left\|\Psi\right\|_{\mathcal{C}^\eta} n^{-3} (\log n)^4.$$

In the case that one of the observables has zero mean, [3, Theorem 7.4] states that $|\operatorname{cor}(\Phi, \Psi, n)| \leq C\gamma_n \left\|\Phi\right\|_{\mathcal{C}^\eta} \left\|\Psi\right\|_{\mathcal{C}^\eta}$. So $|\operatorname{cor}(\Phi, \Psi, n)| \leq C n^{-\beta'} \log n \left\|\Phi\right\|_{\mathcal{C}^\eta} \left\|\Psi\right\|_{\mathcal{C}^\eta}$. Theorem 1 is proved. □

References

[1] R. Artuso, A. Prampolini. *Correlation decay for an intermittent area-preserving map.* Phys. Lett. A 246 (1998), no. 5, 407–411. 58F08 [cited on p. 1, 3]

[2] P. Bálint, S. Gouëzel, *Limit theorems in the stadium billiard.* Comm. Math. Phys. 263 (2006), no. 2, 461–512. [cited on p. 1]

[3] H. Briun, D. Terhesiu, I. Melbourne. *Sharp polynomial bounds on decay of correlations for multidimensional nonuniformly hyperbolic systems and billiards*, Preprint arXiv:1811.07775. [cited on p. 1, 2, 14, 16]

[4] N. Chernov, H.K. Zhang. Billiards with polynomial mixing rates, *Nonlinearity* 18 (2005), 1527–1553. [cited on p.]

[5] P. Eslami, I. Melbourne, S. Vaienti. *Sharp Statistical Properties for a Family of Multidimensional Non-Markovian Nonconformal Intermittent Maps*, Preprint arXiv:1904.03184. [cited on p. 1]

[6] S. Gouëzel. *Sharp polynomial estimates for the decay of correlations.* Israel J. Math. 139 (2004), 29–65. [cited on p. 1]

[7] H. Hu. *Conditions for the existence of SBR measures for "almost Anosov" diffeomorphisms.* Trans. Amer. Math. Soc. 352 (1999), no. 5, 2331–2367. [cited on p. 1]

[8] H. Hu S. Vaienti. *Sandro Absolutely continuous invariant measures for non-uniformly expanding maps.* Ergodic Theory Dynam. Systems 29 (2009), no. 4, 1185–1215. [cited on p. 1]

[9] H. Hu, S. Vaienti. *Lower bounds for the decay of correlations in non-uniformly expanding maps.* Ergodic Theory Dynam. Systems 39 (2019), no. 7, 1936–1970. [cited on p. 1]

[10] X. Zhang, H. Hu. *Polynomial decay of correlations for almost Anosov diffeomorphisms.* Ergodic Theory and Dynamical Systems, 39 (2019), no. 3, 832–864. [cited on p. 1]

[11] C. Liverani, M. Martens. *Convergence to equilibrium for intermittent symplectic maps.* Comm. Math. Phys. 260 (2005), no. 3, 527–556. [cited on p. 1, 2, 3, 4, 5, 9, 14, 15]
[12] C. Liverani; B. Saussol; S. Vaienti. *A probabilistic approach to intermittency.* Ergodic Theory Dynam. Systems 19 (1999), no. 3, 671–685. [cited on p. 1]
[13] C. Liverani, D. Terhesiu, *Mixing for some non-uniformly hyperbolic systems.* Ann. Henri Poincaré **17** (2016), no. 1, 179–226. [cited on p. 1]
[14] C. Liverani; M. P. Wojtkowski. *Ergodicity in Hamiltonian systems. Dynamics reported*, 130–202, Dynam. Report. Expositions Dynam. Systems (N.S.), 4, Springer, Berlin, 1995. [cited on p.]
[15] R. Mañé. Ergodic Theory and Differentiable Dynamics, Springer, Berlin, 1987. [cited on p. 15]
[16] O. Sarig. *Thermodynamic formalism for countable Markov shifts.* Ergodic Theory Dynam. Systems 19 (1999), no. 6, 1565–1593. [cited on p. 1]
[17] O. Sarig. *Subexponential decay of correlations*, Inventiones Mathematicae 150 (2002), 629-653. [cited on p. 1]
[18] Q. Wang, L.S. Young. *Toward a theory of rank one attractors.* Ann. of Math. (2) 167 (2008), no. 2, 349–480. [cited on p. 1]
[19] M. Wojtkowski. *Invariant families of cones and Lyapunov exponents.* Ergodic Theory Dynam. Systems 5 (1985), no. 1, 145–161. [cited on p. 2]
[20] L.S. Young, *Statistical properties of dynamical systems with some hyperbolicity.* Ann. of Math. (2) 147 (1998), no. 3, 585–650. [cited on p. 1, 14, 15]
[21] L.S. Young. *Recurrence times and rates of mixing.* Israel J. Math. 110 (1999), 153–188. [cited on p. 1]
[22] L.S. Young. *What are SRB measures, and which dynamical systems have them? Dedicated to David Ruelle and Yasha Sinai on the occasion of their 65th birthdays.* J. Statist. Phys. 108 (2002), no. 5-6, 733–754. [cited on p.]

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