INDUCING SCHEMES FOR MULTI-DIMENSIONAL PIECEWISE EXPANDING MAPS

PEYMAN ESLAMI

Abstract. We construct inducing schemes for general multi-dimensional piecewise expanding maps where the base transformation is Gibbs-Markov and the return times have exponential tails. Such structures are a crucial tool in proving statistical properties of dynamical systems with some hyperbolicity.

1. Introduction

Statistical properties of chaotic dynamical systems have been a subject of interest for mathematicians and physicists in the past several decades. While such properties are better understood for uniformly hyperbolic systems, the same cannot be said about systems with non-uniform hyperbolicity. The reason is that there are many mechanisms for non-uniform behaviour (e.g. intermittency, existence of critical points or singularities, etc.) and usually they are mixed with regions (or periods) of uniformly hyperbolic behaviour. To treat this difficulty Young [8, 9] proposed an abstract framework to study such systems. She showed that if the system admits a certain structure, called a Young tower, then statistical properties such as rates of decay of correlations can be deduced using the analogy to Markov chains. Since then many other statistical properties have been studied assuming the existence of such structures. However, constructing such structures for various systems is not easy and requires a good understanding of the nature of non-uniformity of hyperbolicity. Even then, it is usually done in a case by case basis.

The purpose of this article is to obtain such structures for general multi-dimensional expanding systems with discontinuities, that is, when the nature of non-uniformity is the presence of discontinuities. This is the first time inducing schemes are constructed in an optimal way (with exponential tails) for general multi-dimensional piecewise expanding maps. As pointed out earlier, virtually every statistical property can then be derived from the existence of such structures (existence and properties of absolutely continuous invariant measures, decay of correlations, central limit theorem, large deviations, Berry-Esseen theorem, almost sure invariance principle, law of iterated logarithm, etc.).

An example of a dynamical system to which this method applies, but previous methods do not, is available in [4]. As shown in [4], the results of the current paper are also relevant to proving statistical properties for non-uniformly expanding dynamical systems. This is because through inducing, one can replace the mechanism of non-uniformity of hyperbolicity with the presence of discontinuities. All of this is worked out for a family of a multidimensional, nonMarkov, nonConformal intermittent maps in [4]. Other systems to which the current framework can be applied are “Hu-Vaienti”-type maps [5] and “Viana”-type maps [7, 1]. Finally, another motivation for this work comes from the study of multi-dimensional dispersing billiards. We refer the reader to the survey [6] for understanding the relevance of this work.

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in providing insights into the problems surrounding the study of multi-dimensional billiards.

The paper is organized in a top-down format in the sense that the main theorems are stated and proved assuming the lemmas and the tools that appear later in the paper.

As the technical details of our proofs may obscure the “big picture” it may help to have the following analogy in mind. Imagine you have a chunk of cookie dough and a few cookie cutters and you want to make cookies. Your main objective is to use all of the dough for this purpose. So you may think that you just expand the dough, use a cookie cutter to cut out your cookies and whatever dough is left over you expand and cut in a similar way until you have finshed the whole dough. But life is not so easy because someone else (the piecewise expanding map a.k.a. the devil) cuts, expands and folds the dough in complicated ways (yet with some restrictions, namely hypotheses (1)–(3)). Our theorems essentially say that you can beat the devil in this game and the proofs are essentially recipes for making cookies at an efficient pace (in a multi-dimensional setting) regardless of what the devil does to complicate the task. A crucial step is to make sure that cookies are cut out in such a way that the left over dough can be reused and that eventually the whole dough is used.

2. Setting and assumptions

Consider $\mathbb{R}^d$ endowed with the Euclidean metric $d$ and the Lebesgue measure $m$. Let $X \subset \mathbb{R}^d$ be a bounded, Borel measurable subset such that

$$\sup_{\varepsilon > 0} \frac{m(\partial_{\varepsilon} X)}{\varepsilon} < \infty,$$

(2.1)

where $\partial_{\varepsilon} X = \text{cl} X \cap \text{cl}(\mathbb{R}^d \setminus X)$ is the topological boundary of $X$ in $\mathbb{R}^d$ and $\partial X = \{x \in X : d(x, \partial X) < \varepsilon\}$. We consider a non-singular piecewise invertible map $T$ on $X$ with respect to the countable partition $\mathcal{P} = \{O_h\}_{h \in \mathcal{H}}$ of open subsets of $X$. This means that $m(X \setminus \bigcup_{h \in \mathcal{H}} O_h) = 0$ and the restrictions $T : O_h \to T(O_h)$ and their inverses are non-singular (i.e. $\forall h \in \mathcal{H}$, $(T|_{O_h})_*(m|_{O_h})$ is equivalent to $m|_{T(O_h)}$) homeomorphisms of $O_h$ onto $T(O_h)$. It is notationally convenient to use $h : T^{-1}O_h \to O_h$ to also denote an inverse branch of $T$ and use $H$ to denote the set of inverse branches of $T$. Accordingly, we denote the set of inverse branches of $T^n$, $n \in \mathbb{N}$, by $\mathcal{H}_n$ and the corresponding partition by $\mathcal{P}_n$. We write $Jh$ for the Radon-Nikodym derivative $d(m \circ h)/dm$.

We make the following assumptions on our dynamical system.

Remark: 1. Note that $X$ is contained in $\text{cl} X$ (closure of $X$), which is the disjoint union of $\text{int} X$ (interior of $X$) and $\partial X$. Moreover, $m(\partial X) = 0$ because of (2.1). So, we can restrict our dynamics to $\text{int} X$, which is an open set. So, without loss of generality, we may assume that $X$ is an open subset of $\mathbb{R}^d$. Consequently, $I \subset X$ is open in $X$ if and only if it is open in $\mathbb{R}^d$.

(1) Uniform expansion: For every $h \in \mathcal{H}$ and $\varepsilon > 0$, denote

$$\Lambda_h(\varepsilon) = \sup_{\{x, y \in T(O_h) : d(x, y) \leq \varepsilon\}} \frac{d(h(x), h(y))}{d(x, y)}.$$ 

There exist $\varepsilon_1 > 0$ and $\Lambda \in (0, 1)$ such that for every $h \in \mathcal{H}$, $\Lambda_h(\varepsilon_1) \leq \Lambda < 1$. Set $\Lambda_h := \Lambda_h(\varepsilon_1)$. Note that for $h \in \mathcal{H}^n$, we can define $\Lambda_h$ using $T^n$ and it is easy to verify that for all $h \in \mathcal{H}^n$, $\Lambda_h(\varepsilon_1) \leq \Lambda^n < 1$. 

(2) Bounded distortion: There exist $\alpha \in (0,1]$, $\hat{D} \geq 0$ such that $\forall h \in H$, $\forall x, y \in T(O_h)$

$$Jh(x) \leq \epsilon \hat{d}(x,y)^\alpha Jh(y). \quad (2.2)$$

Let $D = \hat{D}/(1 - \Lambda^\alpha)$. As a consequence of uniform expansion, (2.2) holds for $h \in H^n$ uniformly for all $n \in \mathbb{N}$ with $D$ instead of $\hat{D}$.

(3) Controlled complexity: There exist $n_0 \in \mathbb{N}$, $\varepsilon_2 > 0$ and $0 \leq \sigma < \Lambda^{-n_0} - 1$ such that for every open set $I$, diam $I \leq \varepsilon_2$, for every $\varepsilon < \varepsilon_2$,

$$\sum_{\{h \in H^n, m(I \cap O_h) > 0\}} \frac{m(h(\partial_h T^n(I \cap O_h)) \setminus \partial_h \Lambda^n I)}{m(\partial_h \Lambda^n I)} \leq \sigma < \Lambda^{-n_0} - 1. \quad (2.3)$$

Moreover, there exists a constant $\bar{C} < \infty$ such that for every integer $1 \leq r < n_0$, for every $\varepsilon < \varepsilon_2$,

$$\sum_{\{h \in H^r, m(I \cap O_h) > 0\}} \frac{m(h(\partial_h T^r(I \cap O_h)) \setminus \partial_h \Lambda^r \varepsilon I)}{m(\partial_h \Lambda^r \varepsilon I)} \leq \bar{C}. \quad (2.4)$$

We refer to the expression on the left-hand side of (2.3) as the complexity expression.

Remark 2. Often one can check (2.3) for $n_0 = 1$ in which case there is no need to check (2.4).

Remark 3. Suppose $h \in H$ and $T|_{O_h} : O_h \to T O_h$ has an extension $\hat{T}_h : \text{cl} O_h \to \text{cl} T O_h$ that is invertible, its inverse $\hat{h}$ satisfies condition (1) and $\partial(T O_h) \subset \hat{T}_h(\partial O_h)$. Then if $A \subset O_h$, we have $\forall \varepsilon < \varepsilon_1$,

$$h(\partial T A) = \hat{h}(\partial T A) = \hat{h}\{y \in T(A) : d(y, \partial T A) < \varepsilon\}$$

$$\subset \{x \in A : d(T x, \hat{T}_h(\partial A)) < \varepsilon\}$$

$$\subset \{x \in A : d(x, \hat{h}(\partial T A)) < \Lambda_h \varepsilon\}$$

$$\subset \{x \in A : d(x, \partial A) < \Lambda_h \varepsilon = \partial_h \varepsilon(A).$$

This is a simple but useful fact to keep in mind when checking (2.3).

Fix

$$a_0 > D/(1 - \Lambda^\alpha) = \hat{D}/(1 - \Lambda^\alpha)^2, \quad \varepsilon_0 = \min\{\varepsilon_1, \varepsilon_2\}. \quad (2.5)$$

3. Statement of the main results

Theorem 1. Suppose $T : X \circ$ satisfies hypotheses (1)-(3). There exist a refinement $P'$ of the partition $P$ into open sets (mod 0) and a function $\tau : X \to \mathbb{Z}^+$ constant on elements of $P'$ such that

(a) The map $G = T : X$ is a Gibbs-Markov map with finitely many images.

(b) $m(\tau > n) \leq \text{const} \cdot \kappa^n$ for some $\kappa \in (0,1)$.

Remark 4. By a Gibbs-Markov map we mean a piecewise expanding map having uniform expansion and bounded distortion.

Under an ergodicity assumption, the induced map can be upgraded to a full-branched Gibbs-Markov map.

Corollary 1. Suppose $T : X \circ$ satisfies hypotheses (1)-(3) and $(T, m)$ is ergodic. There exist an open set $Z \subset X$ and a refinement $P''$ of the partition $P$ into open sets (mod 0) such that $Z$ is a union of elements of $P''$ and there exists a map $\hat{\tau} : Z \to \mathbb{Z}^+$ constant on elements of $P''$ such that

(a) The map $G = T^\circ : Z \circ$ is a full-branched Gibbs-Markov map.

(b) $m(\hat{\tau} > n) \leq \text{const} \cdot \hat{\kappa}^n$ for some $\hat{\kappa} \in (0,1)$. 
Remark 5. It follows from the proof of Corollary 1 that given any open set \( U \subset \mathbb{R}^d \), there exists an open cube inside \( U \) that can be chosen as the set \( Z \).

By [8, 9], if \((T, m)\) is mixing and admits a full-branched inducing scheme as above, then \( \gcd\{n \geq 1 : m(\{\hat{\tau} = n\}) > 0\} = 1 \). The converse is also true, that is, if we find a full-branched inducing scheme, as above, for which the gcd of the return times is equal to 1, then the system is mixing. The next theorem provides such an inducing scheme under extra conditions.

**Definition 1.** We say that \( Z \subset X \) has a nice boundary if there exists a constant \( C_Z > 0 \) such that \( \forall \varepsilon \geq 0, m(\partial_\varepsilon Z) \leq C_Z \varepsilon \) and for every open set \( I \subset X \) containing \( Z \),

\[
m(\partial_\varepsilon (I \setminus \text{cl}Z) \setminus \partial I) \leq C_Z m(\partial I).
\]

**Remark 6.** Many geometric shapes have nice boundaries. For example, sets with piecewise smooth boundaries (no cusps) including rectangles and balls.

**Definition 2.** We say that \( Z \subset X \) is fully recurrent (at times \( \{n_j\}_{j=1}^K \)) if there exist \( K \geq 2 \) positive integers \( \{n_j\}_{j=1}^K \) such that \( \gcd\{n_j\}_{j=1}^K = 1 \) and the following holds: There exist inverse branches \( h_{n_1} \in H^{\nu_1}, h_{n_2} \in H^{\nu_2}, \ldots, h_{n_K} \in H^{\nu_K} \) such that \( O_{h_{n_1}}, \ldots, O_{h_{n_K}} \) are pairwise disjoint and for every \( j = 1, \ldots, K \),

\[
T^{n_j}(O_{h_{n_j}} \cap Z) \supset Z.
\]

**Theorem 2.** Suppose \( T : X \circlearrowleft \) satisfies hypotheses (1)-(3). In addition, suppose that for every \( \delta > 0 \) there exists \( Z \subset X \) of \( \text{diam} \, Z \leq \delta \) that is fully recurrent and has a nice boundary. Then, there exist \( \delta' \) such that \( \forall \delta \leq \delta' \) there exist a refinement \( \mathcal{P}' \) of the partition \( \mathcal{P} \) into open sets (mod 0) such that \( Z = Z(\delta) \) is a union of elements of \( \mathcal{P}' \) and there exists a map \( \hat{\tau} : Z \to Z^+ \) constant on elements of \( \mathcal{P}' \) such that

(a) The map \( \hat{G} = T^{\hat{\tau}} : Z \circlearrowleft \) is a full-branched Gibbs-Markov map.

(b) \( \gcd\{n \geq 1 : m(\{\hat{\tau} = n\}) > 0\} = 1 \).

(c) \( m(\hat{\tau} > n) \leq \text{const} \cdot \hat{\kappa}^n \) for some \( \hat{\kappa} \in (0, 1) \).

**Theorem 3.** Suppose \( T : X \circlearrowleft \) satisfies hypotheses (1)-(3). In addition, suppose that for every \( \delta > 0 \) there exist \( Z \subset Z' \subset X \) such that \( Z \) has a nice boundary, \( \text{diam} \, Z' \leq \delta \), \( m(Z') > m(Z) \) and there exists \( h \in H \) such that \( O_h \subset Z \) and \( TO_h \supset Z' \). Then, there exist \( \delta' \) such that \( \forall \delta \leq \delta' \) there exist a refinement \( \mathcal{P}' \) of the partition \( \mathcal{P} \) into open sets (mod 0) such that \( Z = Z(\delta) \) is a union of elements of \( \mathcal{P}' \) and there exists a map \( \hat{\tau} : Z \to Z^+ \) constant on elements of \( \mathcal{P}' \) such that

(a) The map \( \hat{G} = T^{\hat{\tau}} : Z \circlearrowleft \) is a full-branched Gibbs-Markov map.

(b) \( m(O_h \cap \{\hat{\tau} = 1\}) > 0 \).

(c) \( m(\hat{\tau} > n) \leq \text{const} \cdot \hat{\kappa}^n \) for some \( \hat{\kappa} \in (0, 1) \).

4. **Proof of Theorem 1 and Corollary 1**

Note that \( a_0, \varepsilon_0 \) and \( B_0, \delta_0 \) are constants that depend on the map and are fixed once and for all once \( T \) is fixed. \( a_0, \varepsilon_0 \) are defined in (2.5) and \( B_0, \delta_0 \) are defined in Section 8 in Proposition 1 and Remark 12.

**Proof of Theorem 1.** The following steps lead to our sought after inducing scheme.

(1) Fix \( \delta = \delta_0 \) and consider the partition \( \mathcal{R} \) of \( X \) given by Lemma 2. Let us focus on defining the inducing scheme on one element of this partition. The same can be done for all other partition elements and in a uniform way because \( \mathcal{R} \) is finite. Fix \( R \in \mathcal{R} \) and let \( \mathcal{G}_0 = \{ (R, 1_R/m(R)) \} \) and \( w_0 = m(R) > 0 \). Due to item (2) of Lemma 2, the singleton family \( \mathcal{G}_0 \) with associated weight \( \{w_0\} \) is
an \((a_0, \varepsilon_0, B)\)-proper standard family (Definition 4) for some constant \(B > 0\) possibly larger than \(B_0\).

(2) By Proposition 1, \(\mathcal{G}_1 := \mathcal{T}^{n_{\text{rec}}(B)}\mathcal{G}_0\) is an \((a_0, \varepsilon_0, B_0)\)-proper standard family.

(3) By item (3) of Lemma 2, every standard pair in \(\mathcal{G}_1\) whose domain is \(\delta_0\)-regular contains at least one element \(R'\) from the collection \(\mathcal{R}\). “Stop” each \(\delta_0\)-regular standard pair of \(\mathcal{G}_1\) on its corresponding rectangle \(R' \in \mathcal{R}\). By stopping we mean going back to \(R\) and defining the return time \(\tau = n_{\text{rec}}(B)\) on the subset of \(R\) that maps onto \(R'\) under \(\mathcal{T}^{n_{\text{rec}}(B)}\). By Lemma 5, applied to \(\mathcal{G}_1\), the ratio of the removed weight from \(\mathcal{G}_1\) to the weight of the remainder family (defined in Lemma 3), which we denote by \(\mathcal{G}_1\), is at least some positive constant \(t\) given by \((7.5)\). Since the weight of standard pairs is preserved under iteration, this corresponds to defining \(\tau\) on a subset \(A \subset R\) such that \(m(A) \geq t \cdot m(R \setminus A)\).

Also, by Lemma 3, \(\mathcal{G}_1\) is an \((a_0, \varepsilon_0, \bar{C}_R B_0)\)-standard family.

(4) Just as in step (2), \(\mathcal{G}_2 := \mathcal{T}^{n_{\text{rec}}(C_\ell B_0)}\mathcal{G}_1\) is an \((a_0, \varepsilon_0, B_0)\)-proper standard family so we can apply step (3) to it.

(5) Repeat the steps (3), (4) \(\rightarrow (3), (4) \rightarrow \cdots\), incrementing the indices accordingly during the process.

Applying the above inductive procedure, we will get a “stopping time” (or return time) \(\tau : X \to \mathbb{N}\) defined on a \((\text{mod } 0)\)-partition \(\mathcal{P}'\) of \(X\). \(\mathcal{P}'\) is a refinement of the partition \(\mathcal{P}\) and \(\tau\) is constant on each element of \(\mathcal{P}'\). The return time \(\tau\) will have exponential tails because at each step (where the time between steps is universally bounded by \(n_{\text{rec}}(B) + n_{\text{rec}}(\bar{C}_R B_0)\)) it is defined on a set \(A \subset X\), where \(m(A) \geq t \cdot m(X \setminus A)\). By construction the induced map has finitely many images which form a sub-collection of \(\mathcal{R}\). Note that distortion bound is always maintained under iterations of \(T\) by assumptions (1) and (2) so we need not worry about it.

\(\square\)

Proof of Corollary 1. Let \(Z\) be one of the finitely many images of \(G\). Suppose \(Z\) is minimal in the sense that no proper subsets of \(Z\) is an image of \(G\). Let \(\varsigma : Z \to \mathbb{N}\) be the first return time of \(G\) to \(Z\) and \(\tilde{G} = G^\varsigma : Z \circlearrowleft\) be the associated first return map. Let \(\mathcal{P}''\) be the partition of \(\tilde{G}\), which is a refinement of \(\mathcal{P}'\) and hence of \(\mathcal{P}\). Since \((T, \mathbf{m})\) is ergodic, so is \((G, \mathbf{m})\) hence \(\varsigma\) is well defined. Since \(G\) is Markov and \(Z\) is minimal, \(\tilde{G}\) is full-branched. Define \(\tilde{\tau} = \sum_{\ell=0}^{n-1} \tau \circ G^\ell : Z \to \mathbb{N}\), then \(\tilde{G} = T^{\tilde{\tau}}\).

Since \(G\) is a Markov map with finitely many images, \(\varsigma\) has exponential tails and therefore \(\tilde{\tau} : X \to \mathbb{N}\) also has exponential tails.

\(\square\)

5. Proof of Theorem 2

Proof of Theorem 2. We follow a line of reasoning similar to that of the proof of Theorem 1, but with some modifications when dealing with \(R = Z\) mainly in order to achieve item (b) of Theorem 2.

(1) Fix \(\delta = \delta_0\) and \(c = 1/(2^{d+2}V^d\sqrt{d})\). Let \(Z = Z(c\delta)\) be as in the hypothesis of Theorem 2. Note that \(Z\) is also contained in a \(d\)-dimensional cube of side length \(c\delta\). Let \(\mathcal{R} = \mathcal{R}(\delta)\) be the partition given by Lemma 2. Let \(\mathcal{G}_0 = \{(Z, \mathbb{1}_Z, m(Z))\} \cup w_0 = m(Z)\). \(\mathcal{G}_0\) is an \((a_0, \varepsilon_0, B)\)-proper standard family for some \(B > 0\). Applying Lemma 6 with \(C_1 = n_{\text{rec}}(B)\) and \(C_2 = n_{\text{rec}}(\bar{C}_R B_0)\), it follows that \(Z\) is fully recurrent at times \(\{n_j\}_{j=1}^{K+1}\), where

\[n_1 \geq n_{\text{rec}}(B) \quad \text{and} \quad n_{j+1} - n_j \geq n_{\text{rec}}(\bar{C}_R B_0), \quad \forall j \in \{1, \ldots, K - 1\}\]

(2) Let \(\mathcal{G}_1 := \mathcal{T}^{n_1}\mathcal{G}_0\), taking \(V_\varepsilon = \text{cl} Z\) as the set to avoid under \(\mathcal{T}^{n_1}\) under artificial chopping. This can be done due to Lemma 7. Since \(n_1 \geq n_{\text{rec}}(B)\), \(\mathcal{G}_1\) is an \((a_0, \varepsilon_0, B_0)\)-proper standard family.
(2.1) There exists a standard pair in $G_{1}$ whose domain contains $Z$. “Stop” it on $Z$. That is, define $\tau = n_{1}$ on

$$A_{1} := h_{n_{1}}(Z) \cap Z = \{ x \in O_{h_{n_{1}}} \cap Z : T^{n_{1}}x \in Z \}.$$ 

Note that $T^{n_{1}}A_{1} = Z$. By Lemma 3, the remainder from $G_{1}$, which we denote by $\hat{G}_{1}$ is an $(a_{0}, \varepsilon_{0}, \hat{C}_{R}B_{0})$–proper standard family. Let $G_{2} = T^{n_{1}}-n_{1}\hat{G}_{1}$. Since $n_{2} - n_{1} \geq n_{rec}(\hat{C}_{R}B_{0})$, $G_{2}$ is an $(a_{0}, \varepsilon_{0}, B_{0})$–proper standard family.

(2.2) As before, define $\tau = n_{2}$ on $A_{2} := h_{n_{2}}(Z) \cap Z$. Note that $T^{n_{2}}A_{2} = Z$ and $A_{2}$ is disjoint from $A_{1}$ because $O_{h_{n_{2}}}$ is disjoint from $O_{h_{n_{1}}}$. By Lemma 3, the remainder from $\hat{G}_{2}$, which we denote by $\hat{G}_{2}$ is an $(a_{0}, \varepsilon_{0}, \hat{C}_{R}B_{0})$–proper standard family. Let $G_{3} = T^{n_{2}}-n_{2}\hat{G}_{2}$. Since $n_{3} - n_{2} \geq n_{rec}(\hat{C}_{R}B_{0})$, $G_{3}$ is an $(a_{0}, \varepsilon_{0}, B_{0})$–proper standard family.

(2.3) We continue this process until we define $\tau = n_{K}$ on $A_{K} := h_{n_{K}}(Z) \cap Z$, which is disjoint from previous $A_{j}$’s.

Let $\hat{G}_{K}$ be the remainder from $G_{K}$. Note that $\hat{G}_{K}$ is an $(a_{0}, \varepsilon_{0}, \hat{C}_{R}B_{0})$–proper standard family. Also note that $\forall j \in \{1, \ldots, K\}$, $A_{j} \subset Z$ and $T^{n_{j}}A_{j} = Z$. Moreover, $m(A_{j}) > 0$ because $\forall j \in \{1, \ldots, K\}$ the inverse branches of $T^{n_{j}}$ are non-singular, there are at most countably many such branches and $m(Z) > 0$.

(3) We have achieved that

$$\gcd \left\{ n : m(\{ \tau = n \} \cap \bigcup_{j=1}^{K} A_{j}) > 0 \right\} = 1.$$ 

Also, by construction, $T^{\tau}$ maps each $A_{j}$ onto $Z$ in a one-to-one fashion.

We continue the construction of $\tau$ on the rest of $Z$, i.e. on $\hat{Z} := Z \setminus \bigcup_{j=1}^{K} A_{j}$, in such a way that it has exponential tails. We will do so by continuing to iterate $\hat{G}_{K}$.

(4) Let $G_{K+1} := T^{n_{rec}(\hat{C}_{R}B_{0})}\hat{G}_{K}$. Then $G_{K+1}$ is an $(a_{0}, \varepsilon_{0}, B_{0})$–proper standard family. By item (3) of Lemma 2, every standard pair in $G_{K+1}$ whose domain is $\delta_{0}$–regular contains an element $R_{k}$, $1 \leq k \leq N$, from the collection $\mathcal{R}_{\tau}$. “Stop” such standard pairs of $G_{K+1}$ on $R_{k} \in \mathcal{R}_{\tau}$. By stopping we mean going back to $\hat{Z} \subset Z$ and defining the return time $\tau = n_{K} + n_{rec}(\hat{C}_{R}B_{0})$ on the subset of $\hat{Z}$ that maps onto $R_{k}$ under $T^{n_{K}+n_{rec}(\hat{C}_{R}B_{0})}$. By Lemma 5, the ratio of the removed weight from $G_{K+1}$ to the weight of the remainder family, which we denote by $\hat{G}_{K+1}$, is at least some positive constant $t$ given by (7.5). Note that since the total weight is preserved under iteration, this corresponds to defining $\tau$ on a subset $A \subset \hat{Z}$ such that $m(A) \geq t \cdot m(\hat{Z} \setminus A)$. Also, by Lemma 3, $G_{K+1}$ is an $(a_{0}, \varepsilon_{0}, B_{0})$–proper standard family.

(5) $G_{K+2} := T^{n_{rec}(\hat{C}_{R}B_{0})}\hat{G}_{K+1}$ is an $(a_{0}, \varepsilon_{0}, B_{0})$–proper standard family so we can apply step (4) to it.

(6) Repeat the steps (4), (5) → (4), (5) → ···, incrementing the indices accordingly during the process. This procedure defines $\tau$ on $\hat{Z}$ up to a measure zero set of points (which includes points that map into $\partial \hat{Z}$).

The above steps described how to define $\tau$ on $Z$. We have also explained how to define $\tau$ on the rest of the elements of $\mathcal{R}$ in the proof of Theorem 1. Putting these together we get the same statement as Theorem 1, but with the additional properties that $\gcd \{ n : m(\{ \tau = n \} > 0 \} = 1$, $Z$ is one of the finitely many images of $G = T^{\tau}$ and that $G(Z) \supset Z$. 


Let $\varsigma : Z \to \mathbb{N}$ be the first return time of $G$ to $Z$ and $\tilde{G} = G^\varsigma : Z \cup \{\} \to Z$ be the associated first return map. Since $GA_j = T^\varsigma A_j = T^{\varsigma_0} A_j = Z, \forall j \in \{1, \ldots, K\}$, it follows that $\varsigma = 1$ on the set $\cup_{j=1}^K A_j$.

Define $\tilde{\tau} = \sum_{j=0}^{\varsigma-1} \tau \circ G^j : Z \to \mathbb{N}$, then $\tilde{G} = T^{\tilde{\tau}}$. It follows from the previous paragraph that $\tilde{\tau} = \tau$ on $\cup_{j=1}^K A_j \subset Z$. This implies item (b). Item (a) and item (c) simply follow from the fact that $G$ is a Markov map with finitely many states (hence $\varsigma$ has exponential tails) and $\tau : X \to \mathbb{N}$ has exponential tails. \hfill \Box

6. proof of Theorem 3

The proof of Theorem 3 proceeds similarly to the proof of Theorem 2 except that in the initial step we need to define the stopping time pre-naturally because the initial family is not an $(a_0, \varepsilon_0, B_0)$-standard family and we cannot iterate to make it so. The remedy is to use Lemma 4 where we had previously used Lemma 3.

Proof. (1) Fix $\delta = \delta_0$ and $c = 1/(2^{d+2}Vd\sqrt{d})$. Let $Z = Z(c\delta)$ be as in the hypothesis of Theorem 3. Note that $Z$ is also contained in a $d$-dimensional cube of side length $c\delta$. Let $\mathcal{R} = \mathcal{R}(\delta)$ be the partition given by Lemma 2. Let $\mathcal{G}_0 = \{(Z, 1_Z, m(Z))\}$ and $w_0 = m(Z)$. $\mathcal{G}_0$ is an $(a_0, \varepsilon_0, B)$-proper standard family for some $B > 0$.

(2) Let $\mathcal{G}_1 := T\mathcal{G}_0$, taking $V = c\mathbb{Z}'$ as the set to avoid under $T$ under artificial chopping. This can be done due to Lemma 7. Let $h$ be as in the hypotheses of Theorem 3. Define $\tau = 1$ on

$$A_1 := h(Z) \cap Z = \{x \in O_h \cap Z : T x \in Z\}.$$

Note that $TA_1 = Z$. By Lemma 4, the remainder from $\mathcal{G}_1$, which we denote by $\mathcal{G}_2$, is an $(a_0, \varepsilon_0, B_0)$-proper standard family for some $B_0 > 0$. Let $\bar{Z} := Z \setminus A_1$.

(3) Let $\mathcal{G}_2 := \mathcal{T}^{\nu_{rec}(B')}\mathcal{G}_0$. Then $\mathcal{G}_2$ is an $(a_0, \varepsilon_0, B_0)$-proper standard family.

(4) By item (3) of Lemma 2, every standard pair in $\mathcal{G}_2$ whose domain is $\delta_0$-regular contains at least one element $R'$ from the collection $\mathcal{R}$. “Stop” each $\delta_0$-regular standard pair of $\mathcal{G}_2$ on its corresponding rectangle $R' \in \mathcal{R}$. By stopping we mean going back to $Z$ and defining the return time $\tau = 1 + \nu_{rec}(B')$ on the subset of $R$ that maps onto $R'$ under $\mathcal{T}^{1+\nu_{rec}(B')}$. By Lemma 5, applied to $\mathcal{G}_2$, the ratio of the removed weight from $\mathcal{G}_2$ to the weight of the remainder family (defined in Lemma 3), which we denote by $\tilde{\mathcal{G}_2}$, is at least some positive constant $t$ given by (7.5). Since the weight of standard pairs is preserved under iteration, this corresponds to defining $\tau$ on a subset $A \subset R$ such that $m(A) \geq t \cdot m(R \setminus A)$.

Also, by Lemma 3, $\tilde{\mathcal{G}_2}$ is an $(a_0, \varepsilon_0, C_\mathcal{R}B_0)$-standard family.

(5) Just as in step (3), $\mathcal{G}_3 := \mathcal{T}^{\nu_{rec}(C_\mathcal{R}B_0)}\tilde{\mathcal{G}_2}$ is an $(a_0, \varepsilon_0, B_0)$-proper standard family so we can apply step (4) to it.

(6) Repeat the steps (4), (5) $\rightarrow$ (4), (5) $\rightarrow \cdot\cdot\cdot$, incrementing the indices accordingly during the process.

The above steps described how to define $\tau$ on $Z$ so that $m(O_h \cap \{\tilde{\tau} = 1\}) > 0$. We have also explained how to define $\tau$ on the rest of the elements of $\mathcal{R}$ in the proof of Theorem 1. The rest of the proof is the same as the proof of Theorem 2. \hfill \Box

7. Supplementary lemmas

This section contains supplementary lemmas for the proofs of our main theorems. The first lemma is taken from [2] and stated in a form that is suitable for our needs.

**Lemma 1** (Sublemma C.1 of [2]). Suppose $I$ is a non-empty measurable bounded subset of $\mathbb{R}^d$ and $E$ is a hyperplane cutting $I$ into left and right parts $I_L$ and $I_R$. 


Then \( \forall \varepsilon \geq 0 \) and \( 0 \leq \xi \leq 1 \), we have

\[
m(\{ x \in I : d(x, E) \leq \xi \varepsilon \} \setminus \{ x \in I : d(x, \partial I) \leq \varepsilon \}) \leq \xi m(\{ x \in I : d(x, \partial I) \leq \varepsilon \}).
\]  

(7.1)

**Lemma 2.** There exists a constant \( c > 0 \) such that for every \( \delta > 0 \) and every open \( Z \subset X \) that has a nice boundary and is contained in a \( d \)-dimensional cube of side-length \( c\delta \), there exist a finite (mod 0)-partition \( \mathcal{R} = \{ R_j \}_{j=1}^N \) of \( X \) into open sets such that

1. \( Z \in \mathcal{R} \),
2. for every \( 1 \leq j \leq N \), \( \sup_{x \in \partial R_j} e^{-1} m(\partial_c R_j) < \infty \),
3. for every \( \delta \)-regular set \( I \), there exists \( R \in \mathcal{R} \) s.t. \( I \supset R \) and

\[
\begin{align*}
m(I \setminus R) & \geq (1/2) m(I); \\
m(\partial_c (I \setminus cl R) \setminus \partial_c I) & \leq \max\{2d, C_Z\} m(\partial_c I).
\end{align*}
\]  

(7.2)

(7.3)

Proof. Let \( c = 1/(2^{d+2} V^d \sqrt{d}) \), where \( V = m(\mathcal{B}_1) \) is the volume of the unit ball in \( \mathbb{R}^d \), and let \( S = \{ S_j \} \) denote a grid of open cubes in \( \mathbb{R}^d \) with sides of length \( c\delta \) parallel to the coordinate axes. Since \( X \) is bounded, the collection \( \mathcal{R} = \{ Z \} \cup \{ S \cap (X \setminus Z) : S \in S \} \) forms a finite (mod 0) partition of \( X \) into open sets. Now, if \( R = Z \), then item (2) is satisfied because \( Z \) has a nice boundary. If \( R = S \cap (X \setminus Z) \), then

\[
m(\partial_c R) \leq m(\partial_c S) + m(\partial_c (X \setminus Z)) \\
\leq m(\partial_c S) + m(\partial_c (X \setminus Z)) \setminus \partial_c X + m(\partial_c X).
\]

By a simple calculation \( \leq 2d \varepsilon (c\delta)^{d-1} \). Now item (2) follows because \( Z \) has a nice boundary.

Next, suppose \( I \subset X \) is a \( \delta \)-regular set. Then it contains a ball \( B(x, \delta) \) of radius \( \delta \). Let \( R \in \mathcal{R} \) be the element that contains \( x \). Since \( \text{diam } R \leq \sqrt{d} \varepsilon \delta \),

\[
R \subset B(x, \sqrt{d} \varepsilon \delta) \subset B(x, \frac{1}{2} \delta) \subset I.
\]

Moreover,

\[
m(R) \leq (c\delta)^d = \frac{2^d \delta^d}{V^d} m(B(x, \frac{1}{2} \delta)) < \frac{1}{2} B(x, \frac{1}{2} \delta) \leq \frac{1}{2} m(I).
\]

It follows that \( m(I \setminus R) \geq \frac{1}{2} m(I) \) verifying (7.2). Note that either \( R = Z \) or \( R = S \) for some \( S \in S \). In the former case (7.3) holds because \( Z \) has a nice boundary. In the case that \( R = S \), each of the 2d sides of the cube \( R \) can be continued as a hyperplane to cross \( I \). By Lemma 1, the \( \varepsilon \)-boundary of each side contributes no more than the \( \varepsilon \)-boundary of \( I \), verifying (7.3). \[ \square \]

Set \( c_R = 1/2 \) and \( C_R = 2 \max\{2d, C_Z\} \).

**Lemma 3 (Remainder family \( \mathcal{G}_\delta \)).** Suppose \( \mathcal{G} \) is an \( (a_0, \varepsilon_0, B_0) \)-proper standard family. Let \( \mathcal{G}_\delta \) be the family obtained from \( \mathcal{G} \) by replacing each \( (I, \rho) \) of weight \( w \) having a \( \delta_0 \)-regular domain and containing an element \( R = R(I) \in \mathcal{R} \) in its domain with \( m(I \setminus R) \neq 0 \), by \( (I \setminus \text{cl } R, \rho I \setminus \text{cl } R / \text{cl } R \rho, \rho) \) of weight \( w I \setminus R \rho \). Then \( \mathcal{G}_\delta \) is an \( (a_0, \varepsilon_0, C_R B_0) \)-proper standard family, where \( C_R^{\alpha} = (e^{a_0 \varepsilon_0} C_R + 1)e^{a_0 \varepsilon_0} c_R^{-1} \).
Proof. This is a consequence of item (3) of Lemma 2. Indeed, assuming $\mathcal{G} = \{(I_j, \rho_j)\}$ with associated weights $w_j$, we have, $\forall \varepsilon < \varepsilon_0$,

$$\left| \partial_\varepsilon \mathcal{G} \right| \leq \sum_j w_j \int_{\partial_\varepsilon (I_j \setminus \varepsilon Z)} \rho_j \leq \sum_j w_j \left( \int_{\partial_\varepsilon (I_j \setminus \varepsilon Z) \setminus \partial I_j} \rho_j + \int_{\partial I_j} \rho_j \right)$$

$$\leq \sum_j w_j \left( e^{-\alpha_0 c_0} \frac{\mathbf{m}(\partial_\varepsilon (I_j \setminus \varepsilon Z) \setminus \partial I_j)}{\mathbf{m}(\partial I_j)} \int_{\partial I_j} \rho_j + \int_{\partial I_j} \rho_j \right)$$

$$\leq (e^{-\alpha_0 c_0} C_R + 1) |\partial_\varepsilon \mathcal{G}|,$$

where in the second line we have used the Comparability Lemma 8 and in the last line we have used (7.3). Since $\mathcal{G}$ is $B_0$-proper, $|\partial_\varepsilon \mathcal{G}| \leq B_0 |\mathcal{G}|$; moreover (7.2) can be used to show that $|\mathcal{G}| \leq e^{\alpha_0 c_0} C_R^{-1} |\mathcal{G}|$. Indeed, by Lemma 8,

$$|\mathcal{G}| \geq \sum_j w_j \int_{I_j \setminus \varepsilon Z} \rho_j \geq \sum_j w_j e^{-\alpha_0 c_0} \frac{\mathbf{m}(I_j \setminus \varepsilon Z)}{\mathbf{m}(I_j)} \int_{I_j} \rho_j \geq e^{-\alpha_0 c_0} C_R |\mathcal{G}|.$$

It follows that $|\partial_\varepsilon \mathcal{G}| \leq C_R B_0 |\mathcal{G}|$. \hfill \Box

**Lemma 4** (Remainder family $\mathcal{G}$ in the presence of $Z$). Suppose $\mathcal{G}$ is an $(a_0, \varepsilon_0, B)$-proper standard family. Suppose $Z \subset Z' \subset X$, $Z$ has a nice boundary and $\mathbf{m}(Z') > \mathbf{m}(Z)$. Let $\mathcal{G}$ be the family obtained from $\mathcal{G}$ by replacing each $(I, \rho)$ of weight $w$ containing $Z'$ in its domain, by $(I \setminus \varepsilon Z, \rho_{|\cdot \setminus \varepsilon Z} \int_{I \setminus \varepsilon Z} \rho)$ of weight $w_{|\cdot \setminus \varepsilon Z}$. Then $\mathcal{G}$ is an $(a_0, \varepsilon_0, B')$-proper standard family, for some constant $B' > 0$.

**Proof.** Assuming $\mathcal{G} = \{(I_j, \rho_j)\}$ with associated weights $w_j$, we have, $\forall \varepsilon < \varepsilon_0$,

$$\left| \partial_\varepsilon \mathcal{G} \right| \leq \sum_j w_j \int_{\partial_\varepsilon (I_j \setminus \varepsilon Z)} \rho_j \leq \sum_j w_j \left( \int_{\partial_\varepsilon (I_j \setminus \varepsilon Z) \setminus \partial I_j} \rho_j + \int_{\partial I_j} \rho_j \right)$$

$$\leq \sum_j w_j \left( e^{-\alpha_0 c_0} \frac{\mathbf{m}(\partial_\varepsilon (I_j \setminus \varepsilon Z) \setminus \partial I_j)}{\mathbf{m}(\partial I_j)} \int_{\partial I_j} \rho_j + \int_{\partial I_j} \rho_j \right)$$

$$\leq (e^{-\alpha_0 c_0} C_R + 1) |\partial_\varepsilon \mathcal{G}|,$$

where in the second line we have used the Comparability Lemma 8. In the last line we have used a modified version of (7.3). Note that $I_j$ is not necessarily $\delta_0$-regular, but since $I_j \supset Z$ and $Z$ has a nice boundary the same arguments in the proof of Lemma 2 imply that $\mathbf{m}(\partial_\varepsilon (I \setminus \varepsilon Z) \setminus \partial I_j) \leq C_Z \mathbf{m}(\partial I_j)$.

Since $\mathcal{G}$ is $B$-proper, $|\partial_\varepsilon \mathcal{G}| \leq B_0 |\mathcal{G}|$. By Lemma 8, and since $\mathbf{m}(I_j \setminus Z) \geq \mathbf{m}(Z' \setminus Z) > 0$,

$$|\mathcal{G}| \geq \sum_j w_j \int_{I_j \setminus Z} \rho_j \geq \sum_j w_j e^{-\alpha_0 c_0} \frac{\mathbf{m}(I_j \setminus Z)}{\mathbf{m}(I_j)} \int_{I_j} \rho_j \geq \text{const.} |\mathcal{G}|.$$

It follows that $|\partial_\varepsilon \mathcal{G}| \leq B' |\mathcal{G}|$ for some constant $B' > 0$. \hfill \Box

**Lemma 5.** Let $\mathcal{R} = \{R_k\}_{k=1}^N$ be the partition from Lemma 2. There exists a constant $t > 0$ such that if $\mathcal{G} = \{(I_j, \rho_j)\}_{j \in \mathcal{J}}$ is an $(a_0, \varepsilon_0, B_0)$-proper standard family, then

$$\sum_{j \in \mathcal{J}_{\text{reg}}} w_j \int_{R(I_j)} \rho_j \geq t \cdot \left( \sum_{j \notin \mathcal{J}_{\text{reg}}} w_j + \sum_{j \in \mathcal{J}_{\text{reg}}} w_j \int_{I_j \setminus R(I_j)} \rho_j \right),$$

where $\mathcal{J}_{\text{reg}}$ is the set of $j \in \mathcal{J}$ such that $I_j$ is $\delta_0$-regular.
Proof. Since $\mathcal{G}$ is an $(a_0, \varepsilon_0, B_0)$-proper standard family, at least $2/3$ of its weight is concentrated on $(a_0, \varepsilon_0)$-standard pairs $(I, \rho)$, where $I$ is a $\delta_0$-regular set (recall that $\delta_0 = 1/(3B_0)$). By item (3) of Lemma 2, each such standard pair contains an element from the collection $\mathcal{R}$. Using this fact and the regularity of standard pairs (recall (8.11)), the left-hand side of (7.4) is

$$\geq (2/3)|\mathcal{G}|e^{-a_0 \varepsilon_0^3} m(R(I_j))/m(I_j) \geq (2/3)e^{-a_0 \varepsilon_0^3} C_{ball}(\varepsilon_0)^{-1} m(R(I_j)),$$

where $C_{ball}(\varepsilon_0)$ denotes the measure of a ball of radius $\varepsilon_0$. Now consider the expression in the parentheses and on the right-hand side of (7.4). The first term of this expression is the total weight of the standard pairs that are not $\delta_0$-regular so this term is $\leq (1/3)|\mathcal{G}|$. The second term represents the weights of the remainders, after removing $cl R(I_j)$, from each $\delta_0$-regular $I_j$. This sum is

$$\leq e^{a_0 \varepsilon_0^3} \sum_j w_j m(I_j \setminus R(I_j))/m(I_j) \leq e^{a_0 \varepsilon_0^3} \sum_j w_j m(B_0)/m(R(I_j)) \leq e^{a_0 \varepsilon_0^3} |\mathcal{G}|C_{ball}(\varepsilon_0)/m(\mathcal{R}),$$

where $m(\mathcal{R}) = \min_{1 \leq k \leq N} m(R_k)$ and $B(\varepsilon_0)$ denotes a ball of radius $\varepsilon_0$. So the expression in the parentheses and on the right-hand side of (7.4) is $\leq |\mathcal{G}|(1/3 + e^{a_0 \varepsilon_0^3} C_{ball}(\varepsilon_0)/m(\mathcal{R}))$. Therefore the inequality (7.4) is satisfied if we take:

$$t = \frac{(2/3)|\mathcal{G}|e^{-a_0 \varepsilon_0^3} C_{ball}(\varepsilon_0)^{-1} m(\mathcal{R})}{|\mathcal{G}|(1/3 + e^{a_0 \varepsilon_0^3} C_{ball}(\varepsilon_0)/m(\mathcal{R}))} = \frac{(2/3)e^{-a_0 \varepsilon_0^3} C_{ball}(\varepsilon_0)^{-1} m(\mathcal{R})^2}{(1/3)m(\mathcal{R}) + e^{a_0 \varepsilon_0^3} C_{ball}(\varepsilon_0)}. \quad (7.5)$$

□

Lemma 6. Suppose $Z$ is fully recurrent at times $\{n_j\}_{j=1}^K$ and $C_1, C_2 > 0$ are arbitrary constants. Then $Z$ is fully recurrent at times $\{n_j\}_{j=1}^K$, where $n_1 \geq C_1$ and $n_{j+1} - n_j \geq C_2$ for every $j \in \{1, \ldots, K-1\}$.

Proof. Let $\{m_j\}_{j=1}^K \in \mathbb{N} \cup \{0\}$ be s.t. $m_1 \tilde{n}_K \geq C_1$ and $(m_{j+1} - m_j)\tilde{n}_1 \geq C_2$ and define

$$n_j := \tilde{n}_j + m_j \tilde{n}_K, \text{ if } 1 \leq j \leq K-1;$$

$$n_K := \tilde{n}_K + \sum_{j=1}^{K-1} n_j,$$

It follows from the properties of $\gcd$ that

$$\gcd(n_1, \ldots, n_K) = \gcd(\tilde{n}_1, \ldots, \tilde{n}_K).$$

Also, by definition, $n_1 \geq m_1 \tilde{n}_K \geq C_1$ and $n_{j+1} - n_j \geq (m_{j+1} - m_j)\tilde{n}_1 \geq C_2$.

Since $Z$ covers itself when it returns at times $\{\tilde{n}_j\}$, the same holds at times $\{n_j\}$.

It follows that $Z$ is fully recurrent at times $\{n_j\}_{j=1}^K$. □

8. Toolbox

8.1. Transfer operator. Define the transfer operator $\mathcal{L} : L^1(X, m) \otimes$ as the dual of the Koopman operator $U : L^\infty(X, m) \otimes$, $U g = g \circ T$. By a change of variables, it follows that

$$\mathcal{L} f(x) = \sum_{h \in \mathcal{H}} f \circ h(x) \cdot Jh(x) \cdot 1_{T(0_h)}(x), \text{ for } m\text{-a.e. } x \in X. \quad (8.1)$$

Note that $\mathcal{L}^n f(x) = \sum_{h \in \mathcal{H}_n} f \circ h(x)Jh(x)1_{T^n(0_h)}(x)$, for every $n \in \mathbb{N}$. 

8.2. Standard families. For $\alpha \in (0, 1)$, and a function $\rho : I \to \mathbb{R}^+ := (0, \infty)$, $I \subset X$ define

$$H(\rho) := H_\alpha(\rho) = \sup_{x, y \in I} \frac{|\ln \rho(x) - \ln \rho(y)|}{d(x, y)^\alpha}.$$  \hspace{1cm} (8.2)

**Definition 3** (Standard pair). An $(a, \varepsilon_0)$-standard pair is a pair $(I, \rho)$ consisting of an open set $I \subset X$ and a function $\rho : I \to \mathbb{R}^+$ such that $\text{diam } I \leq \varepsilon_0$, $\int_I \rho = 1$ and

$$H(\rho) \leq a.$$  \hspace{1cm} (8.3)

**Remark 7** (Notation). All integrals where the measure is not indicated are with respect to the underlying measure $\mathfrak{m}$.

**Definition 4** (Standard family). An $(a, \varepsilon_0)$-standard family $\mathcal{G}$ is a set of $(a, \varepsilon_0)$-standard pairs $\{(I_j, \rho_j)\}_{j \in \mathcal{J}}$ and an associated measure $\omega_\mathcal{G}$ on a countable set $\mathcal{J}$. The total weight of a standard family is denoted $|\mathcal{G}| := \sum_{j \in \mathcal{J}} \omega_j$. We say that $\mathcal{G}$ is an $(a, \varepsilon_0, B)$-proper standard family if in addition there exists a constant $B > 0$ such that,

$$|\partial_\mathcal{G}| := \sum_{j \in \mathcal{J}} \omega_j \int_{\partial I_j} \rho_j \leq B |\mathcal{G}| \varepsilon, \text{ for all } \varepsilon < \varepsilon_0.$$  \hspace{1cm} (8.4)

If $\omega_\mathcal{G}$ is a probability measure on $\mathcal{J}$, then $\mathcal{G}$ is called a probability standard family. Note that every $(a, \varepsilon_0)$-standard family induces an absolutely continuous measure on $X$ with the density $\rho_\mathcal{G} := \sum_{j \in \mathcal{J}} \omega_j \rho_j 1_{I_j}$. We say that two standard families $\mathcal{G}$ and $\check{\mathcal{G}}$ are equivalent if $\rho_\mathcal{G} = \rho_{\check{\mathcal{G}}}$.

Next we define what we mean by an iterate of a standard family. Given an $(a, \varepsilon_0)$-standard family $\mathcal{G}$, we define an $n$-th iterate of $\mathcal{G}$ as follows.

**Definition 5** (Iteration). Let $\mathcal{G}$ be an $(a, \varepsilon_0)$-standard family with index set $\mathcal{J}$ and weight $\omega_\mathcal{G}$. For $(j, h) \in \mathcal{J} \times \mathcal{H}^n$ such that $\text{diam } T^n(I_j \cap O_h) > \varepsilon_0$ and for an open set $V_\varepsilon \subset T^n(I_j \cap O_h)$, $\text{diam } V_\varepsilon \leq \varepsilon_0/(4d^{1/2})$ let $U_{(j, h)}$ be the index set of a $(\text{mod } 0)$-partition $\{U_{\ell}\}_{\ell \in U_{(j, h)}}$ of $T^n(I_j \cap O_h)$ into open sets such that

$$\text{diam } U_{\ell} < \varepsilon_0, \forall \ell \in U_{(j, h)},$$  \hspace{1cm} (8.5)

$V_\varepsilon \subset U_{\ell}$ for some $\ell \in U_{(j, h)}$ and such that, setting $V := T^n(I_j \cap O_h)$,

$$\frac{\sum_{\ell \in U_{(j, h)}} \mathfrak{m}(h(\partial \mathcal{U}_\ell \setminus \partial V))}{\mathfrak{m}(h(V))} \leq C_{\varepsilon_0} \varepsilon, \text{ for every } \varepsilon < \varepsilon_0.$$  \hspace{1cm} (8.6)

For $(j, h) \in \mathcal{J} \times \mathcal{H}^n$ such that $\text{diam } T^n(I_j \cap O_h) \leq \varepsilon_0$ set $U_{(j, h)} = \emptyset$. Define

$$\mathcal{J}_n := \{(j, h, \ell) | (j, h) \in \mathcal{J} \times \mathcal{H}^n, \ell \in U_{(j, h)}, \mathfrak{m}(I_j \cap O_h) > 0\}.$$  \hspace{1cm} (8.7)

For every $j_n := (j, h, \ell) \in \mathcal{J}_n$, define $I_{j_n} := T^n(I_j \cap O_h) \cap U_\ell$ and $\rho_{j_n} : I_{j_n} \to \mathbb{R}^+$,

$$\rho_{j_n} := \rho_j \circ h \cdot Jh \cdot z_n^{-1}, \text{ where } z_n := \int_{I_{j_n}} \rho_j \circ h Jh. \text{ Define } T^n \mathcal{G} := \{(I_{j_n}, \rho_{j_n})\}_{j_n \in \mathcal{J}_n} \text{ and associate to it the measure given by}$$

$$w_{T^n \mathcal{G}}(j_n) = z_n \omega_\mathcal{G}(j).$$  \hspace{1cm} (8.8)

**Remark 8** (Notation). To simplify notation throughout the rest of the paper we write $w_{j_n}$ for $w_{T^n \mathcal{G}}(j_n)$ and $w_j$ for $\omega_\mathcal{G}(j)$.

---

\(^1\)The existence of such a partition $\{U_\ell\}$ follows essentially from [3, Proof of Theorem 2.1] and [2, p. 1349], but for the sake of completeness it is also shown in Lemma 7. There may be many admissible choices for such “artificial chopping”. One can make different choices at different iterations hence an $n$-th iterate of $\mathcal{G}$ is by no means uniquely defined (and this does not cause any problems).

\(^2\)When $U_{(j, h)} = \emptyset$, by $(j, h, \ell)$ we mean $(j, h)$. 

Remark 9. If $G$ is an $(\varepsilon_0, \varepsilon_0)$-standard family, then so is $T^n G$ – a fact that is justified by Lemma 9 of the next section. Comparing the definition of the transfer operator applied to a density with the definition of $T^n G$ and the measure associated to it, we see that
\[
L^n \rho_G = \rho_{T^n G}.
\] (8.9)
This is the main connection between the evolution of densities under $L^n$ and the evolution of standard families.

Remark 10. A simple change of variables shows that for every standard family $G$ and every $n \in \mathbb{N}$, $|T^n G| = |G|$. That is, the total weight does not change under iterations. We will make use of this fact throughout the article.

Lemma 7 (Artificial chopping avoiding a small set $V_\ast$). Suppose $V$ is a bounded, open subset of $\mathbb{R}^d$ with $\text{diam} V > \varepsilon_0$, $V_\ast \subset V$ is a subset of $\text{diam} V_\ast \leq \varepsilon_0/(4\sqrt{d})$ and $T$ satisfies (1) and (2). Then, there exists a (mod 0)-partition $\{U_\ell\}_{\ell \in I}$ of $V$ into open sets such that $\text{diam} U_\ell \leq \varepsilon_0 \forall \ell \in I$, $V_\ast \subset U_\ell$ for some $\ell \in I$, and
\[
\sum_{\ell \in I} \mathfrak{m}(\partial (U_\ell \setminus \partial I V)) \leq C_\varepsilon \varepsilon, \text{ for every } \varepsilon < \varepsilon_0,
\] (8.10)
where $C_\varepsilon = e^{D \text{diam}(X)^n} 6d^{3/2} \varepsilon_0^{-1}$.

Proof. \{U_\ell\} will be a family of sets formed by intersecting $V$ with a grid of cubes of side-length $\varepsilon_0/(3\sqrt{d})$. Indeed, following [3, Proof of Theorem 2.1] and [2, p. 1349], let $\varepsilon_0' = \varepsilon_0/(3\sqrt{d})$ and given $0 \leq a_i < \varepsilon_0'$, $i = 1, \ldots, d$, consider the $(d-1)$-dimensional families of hyperplanes:
\[
L_{a_i} = \{x_1, \ldots, x_i, a_i + n_i \varepsilon_0', \ldots, x_{d-1}|n_i \in \mathbb{Z}\}.
\]
Denote the $(d-1)$-dimensional volume of $V \cap L_{a_i}$ by $A_{a_i}$. By Fubini’s theorem, $\int_0^{\varepsilon_0'} a_i \, da_i = \mathfrak{m}(V)$. Therefore, $3a_i'$ such that $A_{a_i'} \leq \mathfrak{m}(V)/\varepsilon_0'$. Let $L = \cup_i L_{a_i'}$ and denote the total $(d-1)$-dimensional volume of $L \cap V$ by $A$. Let $S = \{S_\ell\}_{\ell \in I}$ be the collection of cubes of the grid formed by $L$ that intersect $V$. Let $U_\ell = S_\ell \cap V$, $\forall \ell \in I$. Then we have
\[
\mathfrak{m}(U_{\ell \in I} \setminus \partial(U_\ell \setminus \partial I V)) \leq 2\varepsilon A \leq 2\varepsilon \text{diam}(V)/\varepsilon_0' = 6d^{3/2} \varepsilon \mathfrak{m}(V).
\]
Now (8.10) follows by using the distortion bound (2.2). Finally, suppose $\text{diam} V_\ast \leq \varepsilon_0/(4\sqrt{d})$. Let $\ell' \in I$ be such that $S_{\ell'} \cap V_\ast \neq \emptyset$. Consider the $2d + 2d$ elements of $S$ that share a face or a vertex with the cube $S_{\ell'}$. Denote them by $\{S_{\ell_j}\}_{j=1}^{z+2d+1}$ and let $S_{\ell_\ast} = \cup_{j=1}^{z+2d+1} S_{\ell_j}$, $U_{\ell_\ast} = \cup_{j=1}^{z+2d+1} U_{\ell_j}$. The set $U_{\ell_\ast}$ is contained in a cube of side-length $\varepsilon_0/\sqrt{d}$ hence it has diameter $\leq \varepsilon_0$. Moreover, $U_{\ell_\ast}$ must contain the set $V_\ast$ because otherwise $V_\ast$ would contain one point from $S_{\ell'}$ and another point from $V \setminus S_{\ell_\ast}$. By construction the distance between these points would be greater than $\varepsilon_0/(3\sqrt{d})$, which contradicts $\text{diam} V_\ast \leq \varepsilon_0/(4\sqrt{d})$. Now, in the collection $\{U_\ell\}$, replace the elements $\{U_{\ell_j}\}_{j=1}^{z+2d+1}$ with the set $U_{\ell_\ast}$. Then $V_\ast \subset U_{\ell_\ast}$, $\text{diam} U_{\ell_\ast} \leq \varepsilon_0$ and condition (8.10) is still satisfied.

Next, we show the invariance of standard families under iteration, but first we state a simple lemma that provides a useful consequence of log-Hölder regularity (8.3).

Lemma 9 (Comparability Lemma). If $\rho : I \to \mathbb{R}^+$ satisfies $H(\rho) \leq a$ for some $a \geq 0$ and $\text{diam} I \leq \varepsilon_0$, then for every $J, J' \subset I$ with $\mathfrak{m}(J) \mathfrak{m}(J') \neq 0$,
\[
\inf_{I} \rho \asymp_{a} \mathcal{A}_J \rho \asymp_{a} \mathcal{A}_{J'} \rho \asymp_{a} \sup_{I} \rho,
\] (8.11)
where $A_J \rho = \mathbf{m}(J)^{-1} \int_J \rho$ is the average of $\rho$ on $J$ and $C_1 \simeq a C_2$ means $e^{-a \varepsilon_0} C_1 \leq C_2 \leq e^{a \varepsilon_0} C_1$.

**Proof.** The lemma follows from the fact that if $(I, \rho)$ satisfies $H(\rho) \leq a$, then for every $x, y \in I$,

$$e^{-a d(x,y)^{\alpha}} \rho(y) \leq \rho(x) \leq e^{a d(x,y)^{\alpha}} \rho(y).$$

\[ \square \]

The following lemma together with Definition 5 justify the invariance of an $(a_0, \varepsilon_0)$–standard family under iteration.

**Lemma 9.** Suppose $(I, \rho)$ is an $(a_0, \varepsilon_0)$–standard pair and $(I_n, \rho_n)$ is an image of it under $T^n$ for some $n \in \mathbb{N}$, as in Definition 5. Then $\text{diam}(I_n) \leq \varepsilon_0$, $\int_{I_n} \rho_n = 1$ and

$$H(\rho_n) \leq a_0 (\Lambda^{\alpha n} + a_0^{-1} D).$$

**(8.12)**

**Proof.** Using the definition of $H(\cdot)$, noting its properties under multiplication and composition, and using the expansion of the map, it follows that Let us start by stating

$$H(\rho_n) \leq H(Jh) + \Lambda^{\alpha n} H(\rho_j).$$

By (2.2) we have $H(Jh) \leq D$, and by assumption $H(\rho_j) \leq a_0$, finishing the proof of (8.12).

\[ \square \]

The next lemma plays a crucial role in our arguments because it allows us to control the measure of points that map near the discontinuities.

**Lemma 10 (Growth Lemma).** Suppose $\varepsilon_0 > 0$, $n_0 \in \mathbb{N}$ and $\sigma$ are as in our assumptions. Suppose $G$ is an $(a_0, \varepsilon_0)$–standard family. Then for every $\varepsilon < \varepsilon_0$ we have

$$|\partial x^n G| \leq (1 + e^{a \varepsilon_0} \sigma)|\partial \Lambda^{\alpha n} G| + \zeta_1 |G| \varepsilon,$$

**(8.13)**

where $\zeta_1 = e^{a \varepsilon_0} C_{\varepsilon_0}$.

**Proof.** Suppose $\varepsilon < \varepsilon_0$. We write $n$ for $n_0$. We have, by definition, $|\partial x^n G| = \sum_j w_j \int_{\partial x^n I_j} \rho_j$. We split the sum into two parts according to whether $U_{(j,h)} = \emptyset$ or $U_{(j,h)} \neq \emptyset$. Suppose $U_{(j,h)} = \emptyset$, that is $\text{diam} T^n(I_j \cap O_h) \leq \varepsilon_0$ and $I_j = T^n(I_j \cap O_h)$. By a change of variables,

$$w_j \int_{\partial x^n I_j} \rho_j = w_j \int_{h(\partial x^n I_j)} \rho_j.$$

For every $h \in \mathcal{H}^n$, since $h(\partial x^n I_j) \subset O_h$, we can write

$$h(\partial x^n I_j) \subset (h(\partial x^n I_j) \cap \partial \Lambda^{\alpha n} I_j) \cup (\partial \Lambda^{\alpha n} I_j \cap O_h).$$

**(8.14)**

The integral over $\partial \Lambda^{\alpha n} I_j \cap O_h$, and summed up over $h$ and $j$ is easily estimated by $|\partial \Lambda^{\alpha n} G|$. To estimate the integral of $\rho_j$ over $h(\partial x^n I_j) \cap \partial \Lambda^{\alpha n} I_j$ we compare it, using Lemma 8, to $\int_{\partial \Lambda^{\alpha n} I_j} \rho_j$ and we get

$$\int_{h(\partial x^n I_j) \cap \partial \Lambda^{\alpha n} I_j} \rho_j \leq e^{a \varepsilon_0} \frac{m(h(\partial x^n(I_j \cap O_h)) \setminus \partial \Lambda^{\alpha n} I_j)}{m(\partial \Lambda^{\alpha n} I_j)} \int_{\partial \Lambda^{\alpha n} I_j} \rho_j.$$

Note that if $m(I_j \cap O_h) = 0$, then $m(h(\partial x^n(I_j \cap O_h))) = 0$ since $h(\partial x^n(I_j \cap O_h)) \subset I_j \cap O_h$. By the controlled complexity condition (2.3),

$$\sum_{h \in \mathcal{H}^n} \frac{m(h(\partial x^n(I_j \cap O_h)) \setminus \partial \Lambda^{\alpha n} I_j)}{m(\partial \Lambda^{\alpha n} I_j)} \leq \sigma.$$ 

**(8.15)**

Therefore,

$$\sum_{j \in J} w_j \sum_{h \in \mathcal{H}^n} \int_{h(\partial x^n I_j) \cap \partial \Lambda^{\alpha n} I_j} \rho_j \leq e^{a \varepsilon_0} \sigma |\partial \Lambda^{\alpha n} G|.$$

**(8.16)**
Now suppose that $U_{(j,h)} \neq \emptyset$. By Definition 5, $\sum_{j_n} w_{j_n} \int_{\partial \mathcal{I}_{j_n}} \rho_{j_n}$ is bounded by $\leq \sum_j w_j \int_{\partial \mathcal{I}_{j_0}} \rho_j \circ h J h$. Let us split the integral over two sets. Since $\partial_{\mathcal{I}_{j_n}} \subset U_{\ell}$, we can write

$$\partial_{\mathcal{I}_{j_n}} \subset (\partial_{\mathcal{I}_{j_n}} \setminus \partial \mathcal{T}^n(I_j \cap O_h)) \cup (\partial \mathcal{T}^n(I_j \cap O_h) \cap U_{\ell}). \quad (8.17)$$

Consider the first term on the right-hand side of (8.17). We need to estimate the integral of $\rho_j \circ h J h$ on this set and sum over $\ell, h$ and $j$. Using a change of variables, the integral is

$$\int_{h(\partial_{\mathcal{I}_{j_n}} \setminus \partial \mathcal{T}^n(I_j \cap O_h))} \rho_j.$$

Since $H(p_j) \leq a_0$, $h(\partial_{\mathcal{I}_{j_n}} \setminus \partial \mathcal{T}^n(I_j \cap O_h)) \leq \text{diam}(I_j) \leq \varepsilon_0$ and $\text{diam}(h \mathcal{T}^n(I_j \cap O_h)) = \text{diam}(I_j \cap O_h) \leq \text{diam}(I_j) \leq \varepsilon_0$, we apply Lemma 8 to get

$$\int_{h(\partial_{\mathcal{I}_{j_n}} \setminus \partial \mathcal{T}^n(I_j \cap O_h))} \rho_j \leq e^{\alpha e c_0} \frac{m(h(\partial_{\mathcal{I}_{j_n}} \setminus \partial \mathcal{T}^n(I_j \cap O_h)))}{m(h \mathcal{T}^n(I_j \cap O_h))} \int_{h \mathcal{T}^n(I_j \cap O_h)} \rho_j.$$

Now we sum the above expression over $\ell$, which is implicit in the notation $I_{j_n} = T^n(I_j \cap O_h) \cap U_{\ell}$. Using (8.6), we get

$$\leq e^{\alpha e c_0} C_{c_0} \varepsilon \int_{I_j \cap O_h} \rho_j.$$

Now we sum over $h$, multiply by $w_j$ and sum over $j$. As a result we get the estimate $\leq e^{\alpha e c_0} C_{c_0} \varepsilon |\mathcal{G}|$. Consider the second term on the right-hand side of (8.17). The contribution of this set is equal to $\sum_j w_j \int_{h(\partial_{\mathcal{I}_{j_n}} \setminus \partial \mathcal{T}^n(I_j \cap O_h))} \rho_j$. But this was already included in the upper-bound estimate above (8.14)-(8.16), so we do not need to add it again.

Recall from Section 2 that $n_0$ is such that $\vartheta_1 := \Lambda^n(1 + e^{\alpha e c_0} \sigma) < 1$. Iterating Lemma 10 leads to the following. The proof is standard (uses (2.4)) so we omit it.

**Corollary 2.** There exists $\zeta_2 \geq 0$ such that for every $k \in \mathbb{N}$ and $\varepsilon < \varepsilon_0$,

$$|\partial_T \mathcal{T}^n \mathcal{G}| \leq (1 + e^{\alpha e c_0} \sigma)^k |\partial \Lambda^n \mathcal{G}| + \zeta_2 |\mathcal{G}| \varepsilon. \quad (8.18)$$

Moreover, there exist $\zeta_2, \zeta_3 \geq 0$ such that for every $m \in \mathbb{N}$ that does not divide $n_0$ and for every $\varepsilon < \varepsilon_0$,

$$|\partial_T \mathcal{T}^m \mathcal{G}| \leq \zeta_3(1 + e^{\alpha e c_0} \sigma)^{m/n_0} |\partial \Lambda^n \mathcal{G}| + \zeta_4 |\mathcal{G}| \varepsilon. \quad (8.19)$$

**Proposition 1.** There exists $B_0 > 0$ such that for every $B \geq B_0$ there exists $n_{rec}(B) \in \mathbb{N}$ such that if $\mathcal{G}$ is an $(a_0, \varepsilon_0, B)$-proper standard family, then for every $m \geq n_{rec}(B)$, $\mathcal{T}^m \mathcal{G}$ is an $(a_0, \varepsilon_0, B_0)$-proper standard family.

**Proof.** Choose $B_0 > \zeta_4$. Setting $\vartheta_2 = \vartheta_1^{1/n_0} < 1$ it follows from Corollary 2 that for every $m \in \mathbb{N}$ and $\varepsilon < \varepsilon_0$,

$$|\partial_T \mathcal{T}^m \mathcal{G}| \leq |\mathcal{G}| \varepsilon (B \zeta_3 \vartheta_1^n + \zeta_4). \quad (8.20)$$

Now choose $n_{rec}(B)$ so that $B \zeta_3 \vartheta_1^n + \zeta_4 \leq B_0$. 

**Remark 11.** $n_{rec} : [0, \infty) \to \mathbb{N}$ denotes the time it takes for an $(a_0, \varepsilon_0, -)$-proper standard family to recover to an $(a_0, \varepsilon_0, B_0)$-proper standard family.

**Definition 6.** A set $I \subset X$ is said to be $\delta$-regular if $I$ is open and $\mathfrak{m}(I \setminus \partial \delta I) > 0$.

**Lemma 11.** If $I$ is a $\delta$-regular set, then for every $x \in I \setminus \partial \delta I$, the ball $\mathcal{B}(x, \delta)$ is contained in $I$. 
**Proof.** If $I$ is $\delta$-regular, then $I \setminus \partial \delta I$ is non-empty. Consider a point $x \in I \setminus \partial \delta I$ and the ball (in $\mathbb{R}^d$) $B(x, \delta)$ of radius $\delta$ centered at $x$. If this ball is not entirely contained in $I$, then $B(x, \delta) \cap I$ and $B(x, \delta) \cap (\mathbb{R}^d \setminus \text{cl} I)$ are non-empty open sets in $\mathbb{R}^d$. Furthermore since $I$ is open and $B(x, \delta)$ does not intersect $\partial I$, the union of the sets $B(x, \delta) \cap (\mathbb{R}^d \setminus \text{cl} I)$ is $B(x, \delta)$. This is a contradiction to $B(x, \delta)$ being connected in $\mathbb{R}^d$. □

**Remark 12.** Define $\delta_0 = 1/(3B_0)$. It follows that if $G$ is an $(a_0, \varepsilon_0, B)$-proper standard family, then more than $(2/3)$ of its total weight is concentrated on $\delta_0$-regular sets. That is,

$$\sum_{j \in J_{\text{reg}}} w_j \geq \sum_{j} w_j \int_{I_j \setminus \partial \delta_0 I} \rho_j \geq 2/3,$$

(8.21)

where $J_{\text{reg}}$ corresponds to indices $j$ for which $I_j$ is $\delta_0$-regular.

**References**


Peyman Eslami, Dipartimento di Matematica, II Università di Roma (Tor Vergata), Via della Ricerca Scientifica, 00133 Roma, Italy.

E-mail address: peslami7@gmail.com