ON PIECEWISE EXPANDING MAPS

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Abstract. We study the statistical properties of piecewise expanding maps in the general setting of metric measure spaces. We provide sufficient conditions for exponential mixing of such systems with explicit estimates on the constants. We also provide sufficient conditions for the existence of inducing schemes where the base transformation is Gibbs-Markov and the return times have exponential tails. Such structures can then be used to deduce finer statistical properties.

1. Introduction

In the study of chaotic phenomena, piecewise expanding maps play an important role. Besides being used directly as mathematical models of observed chaotic phenomena, they very often arise in the analysis of other mathematical models, mainly those with “some” hyperbolicity.

The study of piecewise expanding maps has a long history which we will tend to only briefly and selectively based on their relevance to the current work. Most of the earlier results dealt with the existence of absolutely continuous invariant probability measures (ACIPs) for maps of the unit interval. One of the first major results in this direction was obtained by Lasota and Yorke [25] who set up a functional analytic framework and showed that piecewise $C^2$ expanding maps of the unit interval (with a finite partition of monotonicity) admit finitely many ACIPs. Later, using a similar point of view, several authors proved the existence of ACIPs for multi-dimensional piecewise expanding maps under various extra assumptions [23, 19, 1, 31, 11, 15, 32, 29]. The study of piecewise expanding maps in higher dimensions is much more subtle than in dimension one because the geometry of the space and the discontinuities all of a sudden play an important role in the statistical properties of the system. The setting of this paper is more general, hence we also need to deal with such difficulties (and more).

The functional analytic point of view used in the works cited above has proven to be quite fruitful. Strong results on the statistical properties can be obtained once one constructs proper Banach spaces on which to study the spectrum of the transfer operator associated to the dynamical system. However, besides the fact that our setting is considerably more general and that there is no obvious choice of a Banach space adapted to maps on metric spaces, this approach does not lead to explicit estimates in exponential mixing. For example one cannot explicitly estimate the time it takes for an initial density to be within distance $1/2$ of the invariant density – the $1/2$-mixing time. These constants depend on the intrinsic properties of the dynamical system, hence estimating them explicitly requires much better

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quantitative understanding of the obstructions to fast mixing. In particular, the relevant properties must first be extracted and properly quantified. In this article we identify such properties and we use a different approach, that of coupling, in order to obtain explicit estimates of constants that lead to the estimate of mixing times. We also provide an alternative approach for obtaining statistical properties of the system if one is not interested in explicit constants.

Before we comment on the alternative approach, let us briefly comment on our hypotheses and why they are essentially necessary. Our assumption (1) is on uniform (local) expansion. Without this assumption, we are out of the context of piecewise expanding maps, so there is nothing to discuss unless some expansion is present and we can first harvest it by some inducing procedure.

Assumption (2) is only a weak regularity condition expressed through the log-Hölder regularity of the Jacobian of the map (2.1). It may be possible to slightly weaken this condition, but even in the setting of interval maps, it is well-known [22] that $C^1$ regularity of the map (piecewise), is not sufficient for the existence of ACIP’s.

Assumption (3) is a condition that prevents the measure to pile up near singularities. If the boundaries of the pieces on which the map is defined are considered as discontinuities, (3) can be thought of as a condition on the amount of cutting by discontinuities versus the amount of expansion of the map. It has been shown [33, 35, 9] that in some cases high regularity of the map makes up for the need for such an assumption, but in general this assumption is necessary for piecewise expanding maps in dimensions higher than one. For counter-examples when this condition fails we refer the reader to [34, 10]. Note that our assumption (3) is weaker than the usual assumption in comparable works [12, 6] yet we show that the system enjoys good statistical properties under (3). Let us also point out that we allow for our piecewise expanding map to have a countably infinite partition and the space to be non-compact. For maps defined on a countably infinite partition, Rychlik [30] obtained some results for interval maps (using the functional analytic approach), Alves [2] extended the multidimensional result of [19] to a countably infinite partition, and in the non-compact setting Bugiel [8] and Lenci [26] have studied Markov maps of $\mathbb{R}^d$. The method of this paper also allows one to treat non-Markov maps of $\mathbb{R}^d$, which I believe have not been studied before. Indeed, we provide an example of a non-Markov map of $\mathbb{R}$ that fits into our framework.

Assumption (4) is one that is required by our methods because it allows us to study the system locally. It is an assumption on the space in addition to being an assumption on the map and it is satisfied automatically in very general situations for example when the space is a bounded, measurable subset of $\mathbb{R}^n$ and the map satisfies conditions (1) and (2).

Assumption (5), named “positively linked” is only required if one is interested in explicit constants. It basically says that, at a certain fixed scale, all parts of the space communicate with each other under iterations of the map. In dimension one we show how to check this condition by hand, but in higher dimensions it is more difficult and it may be more feasible to check it by developing a computer algorithm. In any case, a condition of this form is necessary and unavoidable for estimating the mixing time of a system.

Results on the rate of mixing for multi-dimensional maps, which also provide explicit estimates on the constants are rare. Saussol [31] obtains such explicit estimates via the approach of Liverani [27] using Hilbert metric contraction; however, in his setting the space is a compact subset of $\mathbb{R}^d$ and he makes assumptions involving the ACIP of the system in order to obtain exponential mixing. We do not make any assumptions on the ACIP of the system (in general one may not have
this information a priori). As far as the method of Hilbert metric contraction is concerned, we will comment on its relation to coupling later in this work.

For the reader who does not care about explicit estimates of constants, we provide various sufficient conditions that lead to powerful inducing schemes that in turn lead to many other statistical properties such as the central limit theorem, large deviations, Berry-Esseen theorem, almost sure invariance principle, law of iterated logarithm, etc. Moreover, such inducing schemes can be combined with other inducing schemes to provide similar information about systems that are not initially piecewise expanding, but can be “induced” to a piecewise expanding map as mentioned earlier.

In my knowledge there are no papers in which general piecewise expanding maps are shown to admit an inducing scheme with exponential tails, not even if the setting of this paper is restricted to maps of the unit interval. The paper [4] (inspired by [3]) provides some conditions under which an inducing scheme with stretched-exponential tails can be obtained; but, for maps with discontinuities, in order to check those conditions, it is necessary (but not nearly sufficient) to do an analysis similar to what is done in this paper.

Finally one motivation for the current work is to eventually prove exponential mixing for certain multi-dimensional chaotic billiards. At the moment there is not even one example of a multi-dimensional chaotic billiard for which exponential mixing is proven. The issue in such systems is known to be the complexity growth of singularities. For a recent survey on the difficulties associated to multi-dimensional billiards we refer the reader to [36]. As pointed out in the last paragraph of section 4.1 of [36], studying piecewise expanding maps can provide valuable intuition on the complexity issue for billiards. Here we show that this issue is resolvable for piecewise expanding map with very general discontinuities (even in the presence of anisotropy of expansion in different directions) providing some hope for progress on the problem of exponential mixing for multi-dimensional billiards.

The essential ingredients of this article are standard families (introduced and developed by D. Dolgopyat and N. Chernov) and the method of coupling (introduced to dynamical systems by L.-S. Young). Both ingredients have been used in various setting by various authors [37, 12, 7, 38, 13, 28, 5, 24], but here they are used in a different way and in a manner more similar to [16].

The outline of the paper is as follows. In Section 2 we describe the assumptions on our dynamical system. Such assumptions were formulated with applications in mind and designed so that they are checkable by considering only finitely many iterates of the map. In Section 3 we define standard pairs and standard families and state our main results on exponential mixing with explicit constants. In section 4 we define the iteration of standard families. In Section 5 we show their invariance under the dynamics and prove the “Growth Lemma”. In Section 6 we describe the coupling of standard families and prove our main theorem. In Section 7, under additional assumptions to those of Section 2, we construct several inducing schemes (Proposition 3, Proposition 4 and Proposition 5) that can be used to deduce various statistical properties of the system under study. The remaining sections are devoted to examples in which we justify that our assumptions are checkable. There is not much that is special about our examples and similar ideas can be applied to check our assumptions for more complicated examples.

2. Setting

Let \((X, d)\) be a metric space, \(\mathcal{B}\) the Borel sigma-algebra and \(m\) a sigma-finite measure on the measurable space \((X, \mathcal{B})\). We assume that \(\exists \epsilon_1 > 0\) s.t. \(\forall \epsilon < \epsilon_1\) \(\exists C_{\text{ball}}(\epsilon) > 0\) s.t. for every open ball \(B\) of diam \(B \leq \epsilon\), \(m(B) \leq C_{\text{ball}}(\epsilon)\).
We consider a non-singular piecewise invertible map $T$ on $X$ with respect to the countable partition $\mathcal{P} = \{O_h\}_{h \in \mathcal{H}}$ of open subsets of $X$. This means that $\mathbf{m}(X \setminus \bigcup_{h \in \mathcal{H}} O_h) = 0$ and the restrictions $T : O_h \to T(O_h)$ and their inverse are non-singular (i.e. $\forall h \in \mathcal{H}, (T|_{O_h})_* (\mathbf{m}|_{O_h})$ is equivalent to $\mathbf{m}|_{T(O_h)}$) homeomorphisms of $O_h$ onto $T(O_h)$. It is notionally convenient to use $h$ to also denote an inverse branch of $T$ and use $\mathcal{H}$ to denote the set of inverse branches of $T$. Accordingly, we denote the set of inverse branches of $T^n$, $n \in \mathbb{N}$, by $\mathcal{H}^n$ and the corresponding partition by $\mathcal{P}^n$. We write $Jh$ for the Radon-Nikodym derivative $d(\mathbf{m} \circ h)/d\mathbf{m}$.

We make the following assumptions on our dynamical system.

1. **Uniform expansion:** For every $h \in \mathcal{H}$ and $\epsilon > 0$, denote
   \[
   \Lambda_h(\epsilon) = \sup_{\{x, y \in T(O_h) : d(x, y) \leq \epsilon\}} \frac{d(h(x), h(y))}{d(x, y)}.
   \]
   There exist $\epsilon_2 > 0$ and $\Lambda \in (0, 1)$ such that for every $h \in \mathcal{H}$, $\Lambda_h(\epsilon_2) \leq \Lambda < 1$. Set $\Lambda_h := \Lambda_h(\epsilon_2)$. Note that for $h \in \mathcal{H}^n$, we can define $\Lambda_h$ using $T^n$ and it is easy to verify that for all $h \in \mathcal{H}^n$, $\Lambda_h(\epsilon_2) \leq \Lambda^n < 1$.

2. **Bounded distortion:** There exist $\alpha \in (0, 1]$, $\bar{D} \geq 0$ and $\epsilon_3 > 0$ such that $\forall h \in \mathcal{H}$, $\forall x, y \in T(O_h)$ satisfying $d(x, y) \leq \epsilon_3$, holds
   \[
   Jh(x) \leq \epsilon \bar{D} d(x, y)^{\alpha} Jh(y).
   \]
   Let $D = \bar{D}/(1 - \Lambda^{\alpha})$. As a consequence of uniform expansion, (2.1) holds for $h \in \mathcal{H}^n$ uniformly for all $n \in \mathbb{N}$ with $D$ instead of $\bar{D}$.

**Definition 1.** ($\epsilon$-boundary) For a set $A \subset X$, let $\partial_{\epsilon} A = \{x \in A : d(x, \partial A) < \epsilon\}$, where $\partial A = \text{cl} A \cap \text{cl}(X \setminus A)$ is \(^1\) the topological boundary of $A$ as a subset of $X$. We define $\partial_{\epsilon} A = \emptyset$ if $\partial A = \emptyset$. It is important to note that $\partial_{\epsilon} A$ is always a subset of $A$.

**Remark 1.** The notion of topological boundary of a set depends on the ambient space (in addition to its topology). Sometimes it may be helpful to consider the boundary of a set with respect to a larger ambient space. For example if $A \subset X \subset \mathbb{R}^2$, one could consider the boundary of $A$ as a subset of $\mathbb{R}^2$ instead of $X$. Checking these assumptions with this notion of boundary will lead to the theorem with the same notion of boundary.

Fix $a_0 > D/(1 - \Lambda^{\alpha}) = \bar{D}/(1 - \Lambda^{\alpha})^2$.

3. **Dynamical complexity:** There exist $n_0 \in \mathbb{N}$, $\epsilon_4 > 0$ and $0 \leq \sigma < \Lambda^{-n_0} - 1$ such that for every open set $I$, $\text{diam } I \leq \epsilon_4$, for every $\epsilon < \epsilon_4$,
   \[
   \sum_{\{h \in \mathcal{H}^{n_0}, \mathbf{m}(I \cap O_h) > 0\}} \frac{\mathbf{m}(h(\partial_{T^{n_0}} I \cap O_h) \setminus \partial_{T^{n_0}} \epsilon I)}{\mathbf{m}(\partial_{T^{n_0}} \epsilon I)} \leq \sigma < \Lambda^{-n_0} - 1. \tag{2.2}
   \]
   Moreover, there exists a constant $\bar{C} < \infty$ such that for every integer $1 \leq r < n_0$, for every $\epsilon < \epsilon_4$,
   \[
   \sum_{\{h \in \mathcal{H}^r, \mathbf{m}(I \cap O_h) > 0\}} \frac{\mathbf{m}(h(\partial_{T^r} I \cap O_h) \setminus \partial_{T^r} \epsilon I)}{\mathbf{m}(\partial_{T^r} \epsilon I)} \leq \bar{C}. \tag{2.3}
   \]
   We refer to the expression on the left-hand side of (2.2) as the complexity expression.

**Remark 2.** This condition may seem difficult to verify at first sight because it requires (2.2) to be checked for every small open set $I$. However, in many situations of interest it can be verified, for example if $X$ is an open subset of $\mathbb{R}^d$ with the

\(^1\)Throughout the paper $\text{cl } A$ denotes closure of the set $A$ in the topology of $(X, d)$. 
Remark 3. Often one can check (2.2) for \( n_0 = 1 \) in which case there is no need to check (2.3).

Remark 4. Suppose \( h \in \mathcal{H} \) and \( T|\mathcal{O}_h : O_h \to TO_h \) has an extension \( \overline{T}_h : \text{cl} O_h \to \) cl\( \text{T}O_h \) that is invertible, its inverse \( h \) satisfies condition (1) and \( \partial(\mathcal{O}_h) \subset \overline{T}_h(\partial\mathcal{O}_h) \).

Then if \( A \subset \mathcal{O}_h \), we have \( \forall \epsilon < \varepsilon_2 \),

\[
\text{h}(\partial_\epsilon TA) = h\{y \in T(A) : d(y, \partial(TA)) < \epsilon\}
\subset \{x \in A : d(Tx, \overline{T}_h(\partial A)) < \epsilon\}
\subset \{x \in A : d(x, \partial A) < \Lambda_\epsilon \varepsilon\} = \partial \Lambda_\epsilon(A).
\]

This is a simple but useful fact to keep in mind when checking (2.2).

**Definition 2.** We say that \( \{A_j\}_j \) is a (mod 0)-partition of \( A \) into open sets if \( \{A_j\}_j \) is countable, its elements are pairwise disjoint, each \( A_j \) is open and of positive \( m \)-measure, and \( m(A \setminus \bigcup_{j \in \mathbb{N}} A_j) = 0 \).

Fix \( \varepsilon_0 \leq \min\{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4\} \) so that \( \sigma < e^{-a_\sigma \varepsilon_0^{1/3}}(A^{-n_0} - 1) \).

(4) **Divisibility of large sets:** There exist \( \eta < 1 \) and \( C_{\varepsilon_0} > 0 \) such that for every open set \( I \) with diam \( I \leq \varepsilon_0 \), every \( h \in \mathcal{H} \) s.t. \( m(I \cap \mathcal{O}_h) > 0 \), diam \( T(I \cap \mathcal{O}_h) > \varepsilon_0 \) and any set \( V_* \subset T(I \cap \mathcal{O}_h) \) of diam \( V_* \leq \varepsilon_0 \), there exists a (mod 0)-partition \( \{U_\ell\}_{\ell \in \mathcal{U}} \) of \( V := T(I \cap \mathcal{O}_h) \) into open sets such that diam \( U_\ell \leq \varepsilon_0 \) \( \forall \ell \in \mathcal{U} \), \( V_* \subset U_\ell \) for some \( \ell \in \mathcal{U} \), and

\[
\sum_{\ell \in \mathcal{U}} m(h(\partial_\ell U_\ell \setminus \partial_\ell V)) \leq C_{\varepsilon_0} \varepsilon, \text{ for every } \varepsilon < \varepsilon_0.
\]

**Remark 5.** As a consequence of bounded distortion, if this condition holds, then it holds for all iterates \( T^n \), \( n \in \mathbb{N} \).

Note that \( \text{diam}(h(V)) = \text{diam}(I \cap \mathcal{O}_h) \leq \varepsilon_0 \) ensures that \( m(h(V)) < \infty \) and we can write

\[
\sum_{\ell \in \mathcal{U}} \frac{m(h(\partial_\ell U_\ell \setminus \partial_\ell V))}{m(h(V))} = \frac{\sum_{\ell \in \mathcal{U}} m(h(\partial_\ell U_\ell \setminus \partial_\ell V))}{\sum_{\ell \in \mathcal{U}} m(h(U_\ell))} \leq \frac{\sup_{\ell \in \mathcal{U}} Jh \frac{m(\partial_\ell U_\ell \setminus \partial_\ell V)}{m(U_\ell)}}{\inf_{\ell \in \mathcal{U}} Jh \frac{m(U_\ell)}{m(U_\ell)}} \leq e^{D_{\mathcal{O}_h}^*} \max_{\ell \in \mathcal{U}} \left\{ \frac{m(\partial_\ell U_\ell \setminus \partial_\ell V)}{m(U_\ell)} \right\},
\]

where the last inequality holds by distortion bound if all \( U_\ell \) have diameter less than \( \varepsilon_0 \). In dimension one \( (X \subset \mathbb{R}, m = \text{Lebesgue}) \), when \( V \) is any open interval (possibly unbounded) it is easy to partition \( V \), mod 0, into open intervals \( \{U_\ell\} \) such that \( \forall \ell, \varepsilon_0/3 < m(U_\ell) \leq \varepsilon_0 \) and that \( m(\partial_\ell U_\ell \setminus \partial_\ell V) \leq 2\varepsilon \). Moreover, if an open set \( V_* \subset V \) of diameter \( \leq \varepsilon_0/3 \) is specified in advance, it is easy to ensure that it is contained in one of the partition elements \( U_\ell \in \{U_\ell\} \). This gives the estimate \( \leq e^{D_{\mathcal{O}_h}^*} 6\varepsilon_0^{-1} \varepsilon \) for (2.4). So we can take \( \eta = 1/3 \) and \( C_{\varepsilon_0} = e^{D_{\mathcal{O}_h}^*} 6\varepsilon_0^{-1} \) to satisfy condition (4).

Suppose \( X \) is a bounded measurable subset of \( \mathbb{R}^d \), \( m = \text{Lebesgue} \) and \( T \) satisfies (1) and a slightly stronger bounded distortion condition where (2.1) is satisfied for all \( h \in \mathcal{H} \) and \( x, y \in TO_h \). In this setting, we claim that (4) holds with \( \eta = 1/6 \), \( C_{\varepsilon_0} = e^{D \text{diam}(X)^n} 6d^{3/2} \cdot \varepsilon_0^{-1} \),
families of hyperplanes: \( \varepsilon \) and \( 6 \).

Denote the \((d-1)\)-dimensional volume of \( V \cap L_{a_i} \) by \( A_{a_i} \). By Fubini’s theorem,
\[
\int_{0}^{\varepsilon_0'} A_{a_i} \, da_i = m(V).
\]
Therefore, \( \exists \varepsilon_0' \) such that \( A_{a_i} \leq m(V)/\varepsilon_0' \). Let \( L = \cup_{a_i} L_{a_i} \) and denote the total \((d-1)\)-dimensional volume of \( L \cap V \) by \( A \). Let \( S = \{S_t\}_{t \in \mathcal{U}} \) be the collection of cubes of the grid formed by \( L \) that intersect \( V \). Let \( U_\ell = S_t \cap V \), \( \forall \ell \in \mathcal{U} \). Then we have
\[
m(\cup_{\ell \in \mathcal{U}} (\partial_\ell U_\ell \setminus \partial_\ell V)) \leq 2\varepsilon A \leq 2\varepsilon m(V)/\varepsilon_0' = 6d^{3/2}m(V).
\]

Now (2.1) follows by using the distortion bound. Finally, suppose \( \text{diam} V_* < \varepsilon_0/6 \).

Remark 6. The Growth Lemma (Lemma 3) as well as its corollaries (Corollary 2 and Proposition 1) may be of interest even if one is not interested in coupling, so it is worth pointing out that conditions (1)-(3) and a simplified version of (4) (namely, the version obtained by removing every statement about \( V_\ell \)) suffice to establish Lemma 3 and its corollaries. In Section 10 we show how to check conditions (1)-(4) for a two-dimensional example.

Remark 7. Our assumptions (1)-(4) imply the Growth Lemma which in turn implies that if \( I \) is an open set satisfying \( m(I) > 0 \), \( \text{diam} I \leq \varepsilon_0 \) and \( \sup_{\ell>0} \varepsilon^{-1} m(\partial_\ell I) < \infty \), then for all \( n \in \mathbb{N} \) sufficiently large and \( \forall \varepsilon < \varepsilon_0 \) holds
\[
m \left( \left\{ x \in I : T^n x \in \bigcup_{h \in \mathbb{N}^n} \partial_\ell (I \cap O_h) \right\} \right) \leq B_0 \varepsilon^q,
\]
with \( q = 1 \). However, as pointed out in [14], one may be interested in examples where the above statement is true only with some \( q > 0 \) strictly less than one. There are ways to weaken conditions (3) and (4) in the spirit of arguments in [14] so that the framework of this paper is applicable to examples in which (2.5) holds with \( q \in (0, 1) \), but we do not pursue this path here.

Definition 3. A set \( I \subset X \) is said to be \( \delta_0 \)-regular if \( I \) is open and \( m(I \setminus \partial_\delta I) > 0 \).

Remark 8. We remark that in certain situations a \( \delta_0 \)-regular set must contain a ball \( B \) of a definite size. For example, if \( (X,d) \) is a metric space in which open balls of radius \( \leq \delta_0 \) are connected and if \( I \) is \( \delta_0 \)-regular, then every open ball of radius \( \delta_0 \) centered at a point of \( I \setminus \partial_\delta I \) is contained in \( I \). Indeed, if \( B(x, \delta_0) \) is a ball centered at \( x \in I \setminus \partial_\delta I \) so that \( B(x, \delta_0) \cap (I \setminus I) = \varnothing \), then \( B(x, \delta_0) \cap I \) and \( B(x, \delta_0) \cap (X \setminus \partial_\delta I) \) would be non-empty open sets whose union is \( B(x, \delta_0) \), which contradicts the ball being connected.

More generally, if \( I \) is \( \delta_0 \)-regular and for every \( x \in I \), \( d(x, \partial I) \leq d(x, I \setminus I) \), then \( I \) contains a ball of radius \( \delta_0 \). Indeed, since \( I \) is \( \delta_0 \)-regular, we can choose \( y \in I \) so
that $d(y, \partial I) \geq \delta_0$. Then $d(y, X \setminus I) \geq \delta_0$ hence the open ball of radius $\delta_0$ centered at $y$ is contained in $I$.

Remark 9. If one changes the notion of boundary by measuring it in a larger ambient space as mentioned in Remark 1, then the notion of $\delta_0$-regular set will also change. For example, if $I \subset X = (0, 1)^2 \subset \mathbb{R}^2$ is a $\delta_0$-regular set with respect to $\mathbb{R}^2$-boundary, then $I$ is forced to contain an $\mathbb{R}^2$-ball of radius $\delta_0$; but if it is $\delta_0$-regular with respect to $X$-boundary, then it is only forced to contain a $\delta_0$ ball in the topology of $X$, which could be a sector of a disk with a right angle and radius $\delta_0$.

**Definition 4.** A set $\Omega \subset X$ is a $C_X$-good overlap set if $m(\Omega) > 0$, $m(\partial \Omega) = 0$, and for every open set $V \subset X$, diam $V \leq \varepsilon_0$ containing $\Omega$ and every $\varepsilon < \varepsilon_0$,

$$m(\partial \Omega \setminus \partial \ast V) + m(\partial \ast V \setminus \partial \Omega) \leq C_X m(\partial \ast V).$$

(2.6)

**Remark 10.** Suppose $X$ contains an open set $V$, diam $V \leq \varepsilon_0$ with empty boundary and $\Omega$ is a $C_X$-good overlap set. Then the right-hand side of the above inequality is zero, so the left-hand side must also be zero. This can happen for example if $\Omega$ and $V \setminus \Omega$ also have empty boundary.

Note that if $X$ is such that all balls of diam $\leq \varepsilon_0$ are connected, then every non-empty open set $V \subset X$ with diam $V \leq \varepsilon_0$ has non-empty boundary (otherwise the open ball of diameter $\varepsilon_0$ containing it can be written as a disjoint union of open sets).

Remark 11. In the case that $X = \mathbb{R}^d$ and $m$ is the Lebesgue measure, every non-empty ball $B$ of diam $B \leq \eta \varepsilon_0$ is a $C_X$-good overlap set with $C_X = 1$. For a proof see Lemma 2.2 and the remark immediately after it in [12].

Let us denote $\delta = \delta_0$.

(5) **Positively linked:** There exist constants $C_X > 0$, $N_\delta \in \mathbb{N}$ with $N_\delta \geq M$, $\Delta_\delta > 0$, $\Gamma_{N_\delta} > 0$ and a collection $Q_{N_\delta}$ whose elements are subsets of elements of $\mathcal{P}^{N_\delta}$, such that the following conditions hold:

- $\delta$-density: Every $\delta$–regular set $\Omega \subset X$ contains an element of $Q_{N_\delta}$.
- Overlapping images: For every $Q, \tilde{Q} \in Q_{N_\delta}$ there exists $N$ with $M \leq N \leq N_\delta$ such that $T^N Q \cap T^N \tilde{Q}$ contains a $C_X$-good overlap set $\Omega$ with $m(\Omega) \geq \Delta_\delta > 0$.
- Note that is a function from $Q_{N_\delta} \times Q_{N_\delta}$ into $\{M, M+1, \ldots, N_\delta\}$.
- Positive weight: For every $Q \in Q_{N_\delta}$, $N \in \mathcal{R}(N(Q, \cdot)) := \text{range of the function } N(Q, \cdot)$, and $h \in \mathcal{H}^N$ with $Q \subset O_h$, holds

$$\inf_{T^N(Q)} Jh \geq \Gamma_{N_\delta}.$$  

(2.7)

Let $\gamma = (1/2)C_{\text{ball}}(\varepsilon_0)^{-2}e^{-2a_0}a_0^\alpha 2^2 \Gamma_{N_\delta}$ and $\gamma_1 = (2/3)\gamma$ as in Lemma 6 and Lemma 7. Let $n_1$ be a positive integer such that $2a_0^{\aleph(n_1)} + D < a_0$ (If $a_0 = 0$, set $n_1 = 0$) and let $n_2 = k_0 n_0$, where $k_0$ is such that $(1 + C_X)\theta_1^{k_0} + \varepsilon_0^2/B_0 < 1$ (Recall that $n_0$ was given by condition (3)).

Set $n = N_\delta + \max\{n_1, n_2\}; C_{\gamma_1} = (1 - \gamma_1)^{-1}; \gamma_2 = (1 - \gamma_1)^{1/n}$.

3. **Statement of the main results**

Before we state our main results we need to define several notions. We define the transfer operator $\mathcal{L} : L^1(X, m) \subset L^\infty(X, m) \subset U g = g \circ T$ by a change of variables, it follows that

$$\mathcal{L} f(x) = \sum_{h \in \mathcal{H}} f \circ h(x) \cdot Jh(x) \cdot \mathbf{1}_{T(h(x))}(x), \text{ for } m\text{-a.e. } x \in X.$$ 

(3.1)
Note that $\mathcal{L}^nf(x) = \sum_{h \in \mathcal{H}} f \circ h(x)Jh(x)1_{T^h(O)}(x)$, for every $n \in \mathbb{N}$.

For $\alpha \in (0,1)$, and a function $\rho : I \to \mathbb{R}^+ := (0,\infty)$, $I \subset X$ define

$$H(\rho) := H_\alpha(\rho) = \sup_{x,y \in I} \frac{|\ln \rho(x) - \ln \rho(y)|}{d(x,y)^\alpha}.$$  \hfill (3.2)

**Remark 12** (Notation). All integrals where the measure is not indicated are with respect to the underlying measure $m$.

**Definition 5** (Standard pair). An $(a,\varepsilon_0)$–standard pair is a pair $(I,\rho)$ consisting of an open set $I \subset X$ and a function $\rho : I \to \mathbb{R}^+$ such that $\text{diam} I \leq \varepsilon_0$, $\int_I \rho = 1$ and

$$H(\rho) \leq a.$$  \hfill (3.3)

**Remark 13**. We do not assume that $I$ is connected.

**Definition 6** (Standard family). An $(a,\varepsilon_0)$–standard family $\mathcal{G}$ is a set of $(a,\varepsilon_0)$–standard pairs $\{(I_j,\rho_j)\}_{j \in J}$ and an associated measure $w_\mathcal{G}$ on a countable set $J$. The total weight of a standard family is denoted $|\mathcal{G}| := \sum_{j \in J} w_j$. We say that $\mathcal{G}$ is an $(a,\varepsilon_0,B)$–proper standard family if in addition there exists a constant $B > 0$ such that,

$$|\partial \mathcal{G}| := \sum_{j \in J} w_\mathcal{G}(j) \int_{\partial I_j} \rho_j \leq B|\mathcal{G}|\varepsilon, \text{ for all } \varepsilon < \varepsilon_0.$$  \hfill (3.4)

If $w_\mathcal{G}$ is a probability measure on $J$, then $\mathcal{G}$ is called a probability standard family. Note that every $(a,\varepsilon_0)$–standard family induces an absolutely continuous measure on $X$ with the density $\rho_\mathcal{G} := \sum_{j \in J} w_j \rho_j 1_{I_j}$. We say that two standard families $\mathcal{G}$ and $\tilde{\mathcal{G}}$ are equivalent if $\rho_\mathcal{G} = \rho_{\tilde{\mathcal{G}}}$.

Now we are ready to state our main theorem.

**Theorem 1.** Let $(X,d,m)$ be a metric measure space and $T : X \to X$ a piecewise expanding map satisfying hypotheses (1)-(5) involving parameters $a_0,\varepsilon_0, B_0$. Then there exist $C > 0, \gamma_2 \in (0,1)$ such that for every two $(a_0,\varepsilon_0,B_0)$–proper standard pairs $(I,\rho)$ and $(\tilde{I},\tilde{\rho})$,

$$\|\mathcal{L}^m \rho - \mathcal{L}^m \tilde{\rho}\|_{L_1} \leq C\gamma_2^m, \text{ for every } m \in \mathbb{N}.$$  \hfill (3.5)

The constants $C$ and $\gamma_2$ are explicitly defined above in Section 2 with $C = 2C_{\gamma_1}$.

As a consequence of Theorem 1 there exists a unique absolutely continuous invariant measure with respect to which $T$ is exponentially mixing.

**Corollary 1.** Let $(X,d,m)$ be a metric measure space and $T : X \to X$ a piecewise expanding map satisfying hypotheses (1)-(5). There exists a unique probability density $\varphi \in L^1(X,d,m)$ such that $\mathcal{L} \varphi = \varphi$. Moreover, there exist $C > 0, \gamma_2 \in (0,1)$ such that for every $(a_0,\varepsilon_0,B_0)$–proper probability standard family $\mathcal{G}$,

$$\|\mathcal{L}^m \rho_\mathcal{G} - \varphi\|_{L_1} \leq C\gamma_2^m, \text{ for every } m \in \mathbb{N}.$$  

The constants $C$ and $\gamma_2$ are explicitly defined above in Section 2 with $C = 2C_{\gamma_1}$.

### 4. Iterations of Standard Families

In this section we define what we mean by an iterate of a standard family. Given an $(a,\varepsilon_0)$–standard family $\mathcal{G}$, we define an $n$-th iterate of $\mathcal{G}$ as follows.

**Definition 7** (Iteration). Let $\mathcal{G}$ be an $(a,\varepsilon_0)$–standard family with index set $J$ and weight $w_\mathcal{G}$. For $(j,h) \in J \times \mathcal{H}^n$ such that $\text{diam} T^m(I_j \cap O_h) > \varepsilon_0$ and for an
open set $V_n \subset T^n(I_j \cap O_h)$, $\mathrm{diam} V_n \leq \eta \varepsilon_0$ let $\mathcal{U}_{(j,h)}$ be the index set of a $2 \pmod{0}$-partition \( \{ U_\ell \}_{\ell \in \mathcal{U}_{(j,h)}} \) of $T^n(I_j \cap O_h)$ into open sets such that

$$\text{diam} U_\ell \leq \varepsilon_0, \forall \ell \in \mathcal{U}_{(j,h)}, \tag{4.1}$$

$V_n \subset U_\ell$ for some $\ell \in \mathcal{U}_{(j,h)}$ and such that, setting $V = T^n(I_j \cap O_h)$,

$$\frac{\sum_{\ell \in \mathcal{U}_{(j,h)}} m(h(\partial I_\ell \setminus \partial V))}{m(h(V))} \leq C \varepsilon, \text{ for every } \varepsilon < \varepsilon_0. \tag{4.2}$$

For $(j, h) \in \mathcal{J} \times \mathcal{H}$ such that $\text{diam} T^n(I_j \cap O_h) \leq \varepsilon_0$ set $\mathcal{U}_{(j,h)} = \emptyset$. Define

$$\mathcal{J}_n := \{(j, h, \ell) | (j, h) \in \mathcal{J} \times \mathcal{H}, \ell \in \mathcal{U}_{(j,h)}, \mathcal{I}(I_j \cap O_h) > 0\}. \tag{4.3}$$

For every $j_n := (j, h, \ell) \in \mathcal{J}_n$, define $I_{j_n} := T^n(I_j \cap O_h) \cap U_\ell$ and $\rho_{j_n} : I_{j_n} \rightarrow \mathbb{R}^+$, $\rho_{j_n} := \rho_j \circ h \cdot Jh \cdot z_{j_n}^{-1}$, where $z_{j_n} := \int_{I_j} \rho_j \circ h \cdot Jh$. Define $\mathcal{T}^n \mathcal{G} := \{(I_{j_n}, \rho_{j_n})\}_{j_n \in \mathcal{J}_n}$ and associate to it the measure given by

$$w_{\mathcal{T}^n \mathcal{G}}(j_n) = z_{j_n} w_{\mathcal{G}}(j). \tag{4.4}$$

Remark 14 (Notation). To simplify notation throughout the rest of the paper we write $w_{j_n}$ for $w_{\mathcal{T}^n \mathcal{G}}(j_n)$ and $w_j$ for $w_{\mathcal{G}}(j)$.

Remark 15. If $\mathcal{G}$ is an $(\alpha_0, \varepsilon_0)$-standard family, then so is $\mathcal{T}^n \mathcal{G}$ - a fact that is justified by Lemma 2 of the next section. Comparing the definition of the transfer operator applied to a density with the definition of $\mathcal{T}^n \mathcal{G}$ and the measure associated to it, we see that

$$\mathcal{L}^n \rho_{\mathcal{G}} = \rho_{\mathcal{T}^n \mathcal{G}}. \tag{4.5}$$

This is the main connection between the evolution of densities under $\mathcal{L}^n$ and the evolution of standard families.

Remark 16. A simple change of variables shows that for every standard family $\mathcal{G}$ and every $n \in \mathbb{N}$, $|\mathcal{T}^n \mathcal{G}| = |\mathcal{G}|$. That is, the total weight does not change under iterations. We will make use of this fact throughout the article.

5. Invariance of Standard Families

In this section we first show that an iterate of a standard family is a standard family, then we go on to prove a growth lemma that provides additional information on the properness of a standard family under iteration. Results of this section do not use the positively-linked assumption (5) and only use assumption (4) in its simplified form mentioned in Remark 6.

Let us start by stating a simple lemma that provides a useful consequence of log-Hölder regularity (3.3).

Lemma 1 (Comparability Lemma). If $\rho : I \rightarrow \mathbb{R}^+$ satisfies $H(\rho) \leq a$ for some $a \geq 0$ and $\text{diam} I \leq \varepsilon_0$, then for every $J, J' \subset I$ with $m(J)m(J') \neq 0$,

$$\inf_I \rho \preceq_a A_{J,J'} \rho \preceq_a A_{J,J'} \rho \preceq_a \sup_I \rho, \tag{5.1}$$

where $A_{J,J'} \rho = m(J)^{-1} \int_J \rho$ is the average of $\rho$ on $J$ and $C_1 \preceq_a C_2$ means $e^{-a \varepsilon_0} C_1 \leq C_2 \leq e^{a \varepsilon_0} C_1$.

\footnote{The existence of such a partition $\{ U_\ell \}$ follows from our assumption (4) on divisibility of large sets. There may be many admissible choices for such “artificial chopping”. One can make different choices at different iterations hence an $n$-th iterate of $\mathcal{G}$ is by no means uniquely defined (and this does not cause any problems).}

\footnote{When $U_{(j,h)} = \emptyset$, by $(j, h, \ell)$ we mean $(j, h)$.}
Proof. The lemma follows from the fact that if \((I, \rho)\) satisfies \(H(\rho) \leq a\), then for every \(x, y \in I\),
\[
e^{-ad(x,y)\alpha} \rho(y) \leq \rho(x) \leq e^{ad(x,y)\alpha} \rho(y).
\]

The following lemma together with Definition 7 justify the invariance of an \((a_0, \varepsilon_0)\)-standard family under iteration.

**Lemma 2.** Suppose \((I, \rho)\) is an \((a_0, \varepsilon_0)\)-standard pair and \((I_n, \rho_n)\) is an image of it under \(T^n\) for some \(n \in \mathbb{N}\), as in Definition 7. Then \(\text{diam}(I_n) \leq \varepsilon_0\), \(\int_{I_n} \rho_n = 1\) and
\[
H(\rho_n) \leq a_0(\Lambda^\alpha + a_0^{-1}D).
\]

**Proof.** Using the definition of \(H(\cdot)\), noting its properties under multiplication and composition, and using the expansion of the map, it follows that
\[
H(\rho_n) \leq H(Jh) + \Lambda^\alpha H(\rho_j).
\]
By (2.1) we have \(H(Jh) \leq D\), and by assumption \(H(\rho_j) \leq a_0\), finishing the proof of (5.2). \(\square\)

**Lemma 3 (Growth Lemma).** Suppose \(\varepsilon_0 > 0\), \(n_0 \in \mathbb{N}\) and \(\sigma\) are as in our assumptions. Suppose \(\mathcal{G}\) is an \((a_0, \varepsilon_0)\)-standard family. Then for every \(\varepsilon < \varepsilon_0\) we have
\[
|\partial_x T^n \mathcal{G}| \leq (1 + e^{a_0\varepsilon^2 C\varepsilon_0})|\partial_{\Lambda^\alpha} \mathcal{G}| + \zeta_1|\mathcal{G}|\varepsilon,
\]
where \(\zeta_1 = e^{a_0\varepsilon^2 C\varepsilon_0}\).

**Proof.** Suppose \(\varepsilon < \varepsilon_0\). We write \(n\) for \(n_0\). We have, by definition,
\[
|\partial_x T^n \mathcal{G}| = \sum_j w_j \int_{\partial_x I_n} \rho \int_{h(\partial_x I_n)} \rho_j.
\]
We split the sum into two parts according to whether \(U(j, h) = \emptyset\) or \(U(j, h) \neq \emptyset\).

Suppose \(U(j, h) = \emptyset\), that is \(\text{diam} T^n(I_j \cap O_h) \leq \varepsilon_0\) and \(I_{jn} = T^n(I_j \cap O_h)\). By a change of variables,
\[
w_j \int_{\partial_x I_{jn}} \rho \int_{h(\partial_x I_{jn})} \rho_j.
\]
For every \(h \in \mathcal{H}\), since \(h(\partial_x I_{jn}) \subset O_h\), we can write
\[
h(\partial_x I_{jn}) \subset (h(\partial_x I_{jn}) \setminus \partial_{\Lambda^\alpha} I_j) \cup (\partial_{\Lambda^\alpha} I_j \cap O_h).
\]
The integral over \(\partial_{\Lambda^\alpha} I_j \cap O_h\) and summed up over \(h\) and \(j\) is easily estimated by \(|\partial_{\Lambda^\alpha} \mathcal{G}|\). To estimate the integral of \(\rho_j\) over \(h(\partial_x I_{jn}) \setminus \partial_{\Lambda^\alpha} I_j\) we compare it, using Lemma 1, to \(\int_{\partial_{\Lambda^\alpha} I_j} \rho_j\) and we get
\[
\int_{h(\partial_x I_{jn}) \setminus \partial_{\Lambda^\alpha} I_j} \rho_j \leq e^{a_0\varepsilon^2 C\varepsilon_0} \int_{\partial_{\Lambda^\alpha} I_j} \rho_j.
\]
Note that if \(m(I_j \cap O_h) = 0\), then \(m(h(\partial_x T^n(I_j \cap O_h))) = 0\) since \(h(\partial_x T^n(I_j \cap O_h)) \subset I_j \cap O_h\). By the dynamical complexity condition (2.2),
\[
\sum_{h \in \mathcal{H}} \frac{m(h(\partial_x T^n(I_j \cap O_h)) \setminus \partial_{\Lambda^\alpha} I_j)}{m(\partial_{\Lambda^\alpha} I_j)} \leq \sigma.
\]
Therefore,
\[
\sum_{j \in J} w_j \sum_{h \in \mathcal{H}} \int_{h(\partial_x I_{jn}) \setminus \partial_{\Lambda^\alpha} I_j} \rho_j \leq e^{a_0\varepsilon^2 C\varepsilon_0} |\partial_{\Lambda^\alpha} \mathcal{G}|.
\]
Now suppose that \( U_{(j,h)} \neq \emptyset \). By Definition 7, \( \sum_{j_n} w_{j_n} \int_{\partial_{j_n}} \rho_{j_n} \) is bounded by \( \leq \sum_j w_j \sum_{\ell_n} \int_{\partial_{j_n}} \rho_j o h_j h \). Let us split the integral over two sets. Since \( \partial_{j_n} \subset U_\ell \), we can write
\[
\partial_{j_n} \subset (\partial_{j_n} \cap \partial T^n(I_j \cap O_h)) \cup (\partial T^n(I_j \cap O_h) \cap U_\ell).
\]
(5.6)
Consider the first term on the right-hand side of (5.6). We need to estimate the integral of \( \rho_j o h_j h \) on this set and sum over \( \ell, h, \) and \( j \). Using a change of variables, the integral is
\[
\int_{h(\partial_{j_n} \cap \partial T^n(I_j \cap O_h))} \rho_j.
\]
Since \( H(\rho_j) \leq a_0 \), \( h(\partial_{j_n} \cap \partial T^n(I_j \cap O_h)) \leq \text{diam}(I_j) \leq \varepsilon_0 \) and \( \text{diam}(h(T^n(I_j \cap O_h))) = \text{diam}(I_j \cap O_h) \leq \text{diam}(I_j) \leq \varepsilon_0 \), we apply Lemma 1 to get
\[
\int_{h(\partial_{j_n} \cap \partial T^n(I_j \cap O_h))} \rho_j \leq e^{an_0} \frac{m(h(\partial_{j_n} \cap \partial T^n(I_j \cap O_h)))}{m(h(T^n(I_j \cap O_h)))} \int_{h(T^n(I_j \cap O_h))} \rho_j.
\]
Now we sum the above expression over \( \ell \), which is implicit in the notation \( I_{jn} = T^n(I_j \cap O_h) \cap U_\ell \). Using (4.2), which is a consequence of (2.4) on divisibility of large sets, we get
\[
\leq e^{an_0} C_{n_0} \varepsilon \int_{I_j \cap O_h} \rho_j.
\]
Now we sum over \( h \), multiply by \( w_j \) and sum over \( j \). As a result we get the estimate \( \leq e^{an_0} C_{n_0} \varepsilon |G| \).

Consider the second term on the right-hand side of (5.6). The contribution of \( m \) from this set is equal to \( \sum_j w_j \sum_n \int_{h(\partial T^n(I_j \cap O_h))} \rho_j. \) But this was already included in the estimate above starting with (5.4), so we do not need to add it again. \( \square \)

Recall from Section 2 that \( n_0 \) is such that \( \Lambda^\alpha(1 + e^{an_0} \sigma) < 1 \). Iterating Lemma 3 leads to the following, where the constants involved where defined in Section 2 right before Definition 3. The proof is standard so we omit it.

**Corollary 2.** For every \( k \in \mathbb{N} \) and \( \varepsilon < \varepsilon_0 \),
\[
|\partial_{T^{kn_0}} G| \leq (1 + e^{an_0} \sigma)^k |\partial_{T^{k+\varepsilon}} G| \leq \zeta_2 |G| \varepsilon.
\]
Moreover, for every \( m \in \mathbb{N} \) that does not divide \( n_0 \) and for every \( \varepsilon < \varepsilon_0 \),
\[
|\partial_{T^m} G| \leq \zeta_3 (1 + e^{an_0} \sigma)^m |\partial_{T^{\varepsilon}} G| \leq \zeta_4 |G| \varepsilon.
\]
(5.8)

The following is a direct consequence of Corollary 2 and justifies the fact that the image under \( T^m \) of an \( (a_0, \varepsilon_0, B_0) \)-proper standard family is again an \( (a_0, \varepsilon_0, B_0) \)-proper standard family provided \( m \) is sufficiently large.

**Proposition 1.** Suppose \( G \) is an \( (a_0, \varepsilon_0, B_0) \)-proper standard family. Then for every \( m \in \mathbb{N} \) with \( m/n_0 \in \mathbb{N} \) and every \( \varepsilon < \varepsilon_0 \),
\[
|\partial_{T^m} G| \leq B_0 |G| \varepsilon (\sigma_2^{m/n_0} + \zeta_2 / B_0).
\]
(5.9)
Set \( \sigma_2 = \sigma_1^{1/n_0} \). For every \( m \in \mathbb{N} \) and \( \varepsilon < \varepsilon_0 \),
\[
|\partial_{T^m} G| \leq B_0 |G| \varepsilon (\zeta_3 \sigma_2^m + \zeta_4 / B_0).
\]
(5.10)

By our choice of \( M \) and \( B_0 \) from Section 2 it follows that for every \( m \geq M \) and \( \varepsilon < \varepsilon_0 \), \( |\partial_{T^m} G| \leq B_0 |G| \varepsilon \). So for \( m \geq M \), \( T^m G \) is an \( (a_0, \varepsilon_0, B_0) \)-proper standard family.
Remark 17. Let us record a simple consequence of Corollary 2 for later use. Suppose \( \mathcal{G} \) is an \((a_0, \varepsilon_0, B)\)-proper standard family for some \( B > 0 \). Then it is easy to see from (5.7) that \( \exists n_{rec}(B) \in \mathbb{N} \) such that \( T^{n_{rec}(B)} \) is an \((a_0, \varepsilon_0, B_0)\)-proper standard family. In this way we define \( n_{rec} : [0, \infty) \to \mathbb{N} \) to be the time it takes for an \((a_0, \varepsilon_0, B)\)-proper standard family to recover to an \((a_0, \varepsilon_0, B_0)\)-proper standard family.

6. Coupling

In the previous section we justified the viewpoint of iterating standard families. Now we proceed to explain an inductive procedure to “couple” a small amount of mass of two proper standard families after a fixed number of iterations. It is in this section that we need assumption (5). The reader who is only interested in inducing schemes can safely skip this section. During the coupling procedure standard families are modified in a controlled way. The following two lemmas are related to such modifications.

Lemma 4 (Splitting into constant and remainder). Consider two \((a_0, \varepsilon_0)\)-singleton standard families \( \mathcal{G}_1 = \{(I_1, \rho_1)\} \) and \( \mathcal{G}_2 = \{(I_2, \rho_2)\} \) with associated weights \( w_{1}, w_{2} > 0 \). Suppose \( w_{2} \leq w_{1} \) and let \( c = (1/2)C_{\mathcal{G}_0} e^{-a_0 \varepsilon_0} \). Define

\[
\rho_1 = \frac{w_2}{w_1} c \int_{I_1} \frac{w_2}{w_1} e = \frac{1}{m(I_1)}, \quad \hat{\rho}_1 = (\rho_1 - \frac{w_2}{w_1} c) \int_{I_1} (\rho_1 - \frac{w_2}{w_1} c),
\]

\[
\rho_2 = c \int_{I_2} \frac{c}{c} = \frac{1}{m(I_2)}, \quad \hat{\rho}_2 = (\rho_2 - c) \int_{I_2} (\rho_2 - c). \tag{6.1}
\]

Set \( \breve{w}_1 = w_1 \int_{I_1} (w_2/w_1)c = cw_2m(I_1) \), \( \breve{w}_1 = w_1 \int_{I_1} (\rho_1 - (w_2/w_1)c) \) and \( \breve{w}_2 = w_2 \int_{I_2} (\rho_2 - c) \).

Then \( H(\hat{\rho}_{1,2}) \leq a_0 \), \( H(\hat{\rho}_{1,2}) \leq 2a_0 \) and the \((2a_0, \varepsilon_0)\)-standard families \( \{(I_1, \rho_1), (I_1, \rho_1)\}, \{(I_2, \rho_2), (I_2, \rho_2)\} \) with their associated weights \( \{\breve{w}_1, \breve{w}_1\}, \{\breve{w}_2, \breve{w}_2\} \) are equivalent to \( \mathcal{G}_1 \) and \( \mathcal{G}_2 \), respectively.

Proof. The functions \( \rho_{1,2} \) are constants, so they clearly satisfy \( H(\hat{\rho}_{1,2}) \leq a_0 \) on their domains. Let us show that \( \hat{\rho}_2 \) satisfies \( H(\hat{\rho}_{1,2}) \leq 2a_0 \). Indeed, since \( c \) is chosen such that \( \inf \rho \geq 2c \), we have

\[
\frac{\rho_2(x) - c}{\rho_2(y) - c} \leq 1 + \frac{\rho_2(x) - \rho_2(y)}{\rho_2(y) - c} \leq 1 + 2 \frac{\rho_2(x) - \rho_2(y)}{\rho_2(y)}.
\]

Using \( H(\hat{\rho}_2) \leq a_0 \), the right hand side is further bounded by \( 1 + 2(e^{a_0|x-y|'} - 1) \leq e^{2a_0|x-y|'} \), if \( \rho_2(x)/\rho_2(y) > 1 \) and by \( 1 + 2(1 - e^{-a_0|x-y|''}) \leq 1 + 2(1 - e^{-a_0|x-y|''}) \leq e^{2a_0|x-y|''} \) if \( \rho_2(x)/\rho_2(y) \leq 1 \). As for \( \rho_1 \), it also satisfies \( H(\hat{\rho}_1) \leq 2a_0 \) for the same reason since \( w_2/w_1 \leq 1 \).

To check the equivalence of \( \mathcal{G}_1 \) and the \((2a_0, \varepsilon_0)\)-standard family \( \{(I_1, \rho_1), (I_1, \rho_1)\} \) with associated weights \( \{\breve{w}_1, \breve{w}_1\} \), we check that \( w_1 \rho_1 = \breve{w}_1 \rho_1 + \breve{w}_1 \rho_1 \). Indeed, by construction, \( w_1 \rho_1 = cw_2 \) and \( \breve{w}_1 \rho_1 = w_1 (\rho_1 - (w_2/w_1)c) = w_1 \rho_1 - w_2 c \). Hence the sum is \( w_1 \rho_1 \). The equivalence of \( \mathcal{G}_2 \) to the corresponding \((2a_0, \varepsilon_0)\)-standard family \( \{(I_2, \rho_2), (I_2, \rho_2)\} \) with associated weights \( \{\breve{w}_2, \breve{w}_2\} \) is also easy to check. \( \square \)

Lemma 5 (Chopping out the overlap). Consider two singleton \((a_0, \varepsilon_0)\)-standard families \( \mathcal{G} = \{(I, \rho)\} \) and \( \mathcal{G}_0 = \{(\hat{I}, \hat{\rho})\} \) with associated weights \( w, \hat{w} \). Suppose that \( I \cap \hat{I} \) contains a \( C_X \)-good overlap set \( \Omega \) as defined in Definition 4. Denote \( A_0 = \hat{A}_0 = \emptyset \), \( A_1 = I \cap \hat{I} \) and \( \hat{A}_1 = \hat{I} \setminus \Omega \). Note that the latter two sets can be empty.

There exists an \((a_0, \varepsilon_0)\)-standard family equivalent to \( \mathcal{G} \) obtained by replacing \( \{(I, \rho)\} \) by \( \{\{A_j, 1/m(A_j)\}\}_{j=0} \) and associated weights \( \{cw m(A_j)\}_{j=0} \).\(^4\) Similarly

\(^4\)with the convention that if \( A_1 \) is empty, then we do not include it in the collection.
there exists an \((a_0, \varepsilon_0)\)-standard family equivalent to \(\tilde{G}\) obtained by replacing \([(I, \tilde{c})]\) by \([(A_j, 1/m(A_j))]\) and associated weights \(\{cw(\tilde{A}_j)\}\). Note that if \(cw = \tilde{c}w\), then \(cw(A_0) = \tilde{c}w(A_0)\).

Proof. To show the existence of an \((a_0, \varepsilon_0)\)-standard family equivalent to \(G\) obtained by replacing \([(I, c)]\) by \([(A_j, 1/m(A_j))]\) and associated weights \(\{cw(A_j)\}\), we only need to show that each element of \([(A_j, 1/m(A_j))]\) is an \((a_0, \varepsilon_0)\)-standard pair.\(^5\) By Definition 4, \(\text{diam} \leq \varepsilon_0\) hence \((\Omega, 1/m)\) with associated weight \(cw(\tilde{m}(\Omega))\) is an \((a_0, \varepsilon_0)\)-standard pair.\(^6\) The set \(I \setminus \text{cl} \Omega\) is also open and since \(\text{diam} I \leq \varepsilon_0\), \(\text{diam}(I \setminus \text{cl} \Omega) \leq \varepsilon_0\). Therefore, \((I \setminus \text{cl} \Omega, 1/m(I \setminus \text{cl} \Omega))\) is also an \((a_0, \varepsilon_0)\)-standard pair. Similar statements hold about \(\tilde{G}\). Note that \(m(\text{cl} \Omega) = m(\Omega)\) and similarly for \(I \setminus \text{cl} \Omega\) because \(\tilde{m}(\partial \Omega) = 0\).

We are ready now to couple a small amount of weight of two \((a_0, \varepsilon_0)\)-standard pairs.

Lemma 6. Suppose \(G = [(I, \rho)]\) and \(\tilde{G} = [(I, \tilde{\rho})]\) are singleton \((a_0, \varepsilon_0)\)-standard families with \(\delta\)-regular domains \(I, \tilde{I}\). Let \(N_\delta, \Delta_\delta\) and \(\Gamma_\delta\) be as in the assumptions. There exist \((2a_0, \varepsilon_0)\)-standard families \(G_N, \tilde{G}_N\) such that

\[
\rho \tilde{G}_N - \rho G_N = \rho \Gamma_N G - \rho \Gamma_N \tilde{G}; \quad \text{and},
\]

\[
|G_N| \leq \tilde{|G|} - \min\{|G|, |\tilde{G}|\} \gamma, \quad \text{(6.2)}
\]

\[
|\tilde{G}_N| \leq |\tilde{G}| - \min\{|G|, |\tilde{G}|\} \gamma,
\]

where \(\gamma = (1/2)C_{\tilde{b}a}(\varepsilon_0) - 2e^{-2\alpha^2 e^2} \Delta^2 \Gamma_\delta\).

Proof. Since the sets \(I, \tilde{I}\) are regular sets, by the positively-linked assumption (namely \(\delta\)-density), they each contain an element of \(Q_N\), namely \(Q, \tilde{Q}\). Moreover, there exists \(N\) with \(M \leq N \leq N_\delta\) such that the \((a_0, \varepsilon_0)\)-standard families \(G_N\) and \(\tilde{G}_N\) contain \((a_0, \varepsilon_0)\)-standard pairs \((I_1, \rho_1)\) and \((I_2, \rho_2)\) with associated weights \(w_1, w_2\) whose intersection contains a \(C_X\)-good overlap set \(\Omega\), with \(m(\Omega) \geq \Delta_\delta > 0\). Here we have also used the fact that the artificial chopping of Definition 7 is done avoiding the overlap \(\Omega\).

Let us write \(\Delta = \Delta_\delta\). We assume without loss of generality that \(w_2 = \min\{w_1, w_2\}\). Apply Lemma 4 to replace the \((a_0, \varepsilon_0)\)-standard pairs \((I_1, \rho_1), (I_2, \rho_2)\) by \([(I_1, \tilde{\rho}_1)], [(I_2, \rho_2)]\) with associated weights \(\{\tilde{w}_1, \tilde{w}_1\}, \{\tilde{w}_2, \tilde{w}_2\}\).

Now consider just \((I_1, \tilde{\rho}_1)\) and \((I_2, \rho_2)\). These are constant \((a_0, \varepsilon_0)\)-standard pairs and by definition (see Lemma 4) they satisfy \(\tilde{w}_1 \rho_1 = \tilde{w}_2 \rho_2\). Now apply Lemma 5 to further replace these \((a_0, \varepsilon_0)\)-standard pairs by \([(A_j, 1/m(A_j))]\) with associated weights \(\{\rho_1 \tilde{w}_1 m(A_j), \rho_2 \tilde{w}_2 m(A_j)\}\).

Note that \(A_0 = A_0\) and \(\rho_1 \tilde{w}_1 = \rho_2 \tilde{w}_2 = cw_2\), so the elements corresponding to \(j = 0\) are exactly the same in both families.

At this point we have replaced \(T^N G\) by the \((2a_0, \varepsilon_0)\)-standard family

\[
(T^N G) \setminus \{(I_1, \rho_1)\} \cup \{(I_1, \tilde{\rho}_1)\} \cup \{(A_j, 1/m(A_j))\},
\]

where \(w_1 \rho_1 = \tilde{w}_1 \rho_1 + \sum_{j=0}^{\infty} (\rho_1 \tilde{w}_1 m(A_j) / m(A_j))\).

To complete the modification of our \((a_0, \varepsilon_0)\)-standard families and obtain \((2a_0, \varepsilon_0)\)-standard families \(G_N, \tilde{G}_N\), we remove the common element \((A_0, 1/m(A_0))\) from both collections.

\(^5\)The statement about equivalence is a consequence of \(cw \mathbb{I}_I = cw \mathbb{I}_{A_0} + cw \mathbb{I}_{A_1}\).

\(^6\)To be precise, we should not call \((\Omega, 1/m(\Omega))\) an \((a_0, \varepsilon_0)\)-standard pair because \(\Omega\) is not necessarily open. However, we can afford this abuse of language since \((\Omega, 1/m(\Omega))\) will be removed from standard families during coupling.
The weight of the removed element is \( \tilde{\rho}_i \tilde{w}_i \mathbf{m}(A_0) = c \min\{w_1, w_2\} \mathbf{m}(A_0) \), which by definition of \( c \) (from Lemma 4) and \( \mathbf{m}(A_0) = \mathbf{m}(\Omega) \geq \Delta_i \), is bounded by \( (1/2)C_{B_m}^{-1} e^{-\omega_0 c} \Delta w_2 \). Recall that \( w_2 \) is the weight of \((I_2, \rho_2)\), which is an \((a_0, \varepsilon_0)\)-standard pair in \( T_N \tilde{G} \). Hence for some \( h_2 \in \mathcal{H}^N \), denoting \( w_\mathcal{G} = |\mathcal{G}| \), we have

\[
w_2 = w_\mathcal{G} \int I_2 \tilde{\rho} \circ h_2 J_2 \geq w_\mathcal{G} \int I_\Omega \tilde{\rho} \circ h_2 J_2 \geq w_\mathcal{G} \mathbf{m}(\Omega) \inf \frac{T^{\langle \frac{\Gamma}{\rho} \rangle} J_2 \inf \tilde{\rho}}{I} \geq w_\mathcal{G} \Delta_i \mathbf{m}(\tilde{I})^{-1} \int I \tilde{\rho}.
\]

(6.3)

Since \( \int I \tilde{\rho} = 1 \), we have \( w_2 \geq \Delta_i \mathbf{m}(\tilde{I})^{-1} \). Therefore, the weight of the removed element \((A_0, 1/\mathbf{m}(A_0))\) is bounded by \( \geq (1/2)C_{ball}(\varepsilon_0)^{-2} e^{-2\omega_0 c} \Delta^2 \Gamma_N \min\{|\mathcal{G}|, |\mathcal{G}|\} \). Setting \( \gamma = (1/2)C_{ball}(\varepsilon_0)^{-2} e^{-2\omega_0 c} \Delta^2 \Gamma_N \gamma \),

\[
|\mathcal{G}^*_N| \leq |\mathcal{G}| - \min\{|\mathcal{G}|, |\mathcal{G}|\} \gamma.
\]

One also gets a similar estimate for \( |\mathcal{G}^{\gamma}||\mathcal{N}_N| \).

Now we remove the restriction that \( \mathcal{G} \) and \( \mathcal{G}^\gamma \) are singleton \((a_0, \varepsilon_0)\)-standard families.

**Lemma 7.** Suppose \( \mathcal{G} \) and \( \tilde{\mathcal{G}} \) are \((a_0, \varepsilon_0)\)-standard families and each satisfy \(|\partial \mathcal{G}| \leq B_0|\mathcal{G}| \) \( \varepsilon < \mathcal{G} \). There exist \((2a_0, \varepsilon_0)\)-standard families \( \mathcal{G}^*_N \), \( \mathcal{G}^{\gamma} \) such that \( \rho_{\mathcal{G}^*_N} = Q_{\mathcal{G}^{\gamma}} \) and \( \mathcal{G}^{\gamma} \) such that

\[
\rho_{\mathcal{G}^*_N} - \rho_{\mathcal{G}^{\gamma}} = \rho_{T^N \mathcal{G}} - \rho_{T^N \mathcal{G}^\gamma} \; \text{and},
\]

\[
|\mathcal{G}^*_N| \leq |\mathcal{G}| - \min\{|\mathcal{G}|, |\mathcal{G}|\} \gamma_1,
\]

\[
|\mathcal{G}^{\gamma}| \leq |\mathcal{G}| - \min\{|\mathcal{G}|, |\mathcal{G}|\} \gamma_1,
\]

where \( \gamma_1 = (2/3) \gamma \).

**Proof.** Recall that we chose \( \delta = \delta_0 = 1/(3B_0) \) so that \( |\partial \mathcal{G}| \leq B_0|\mathcal{G}| \delta < (1/3)|\mathcal{G}| \). Let \( \mathcal{G}_{i+1} \subset \mathcal{G} \) be the collection of \((a_0, \varepsilon_0)\)-standard pairs \((I, \rho) \) \in \( \mathcal{G} \) such that \( \int_{\partial I \rho}^1 > 0 \). Note that for such standard pairs, \( I \) is necessarily a \( \delta \)-regular set. We have \( |\mathcal{G}_i| \geq (2/3)|\mathcal{G}| \). Let \( \mathcal{G}_{1} = \mathcal{G} \setminus \mathcal{G}_i \).

Let \( T^N \mathcal{G}_{i+1} \) be an \( N \)-th iterate of \( \mathcal{G}_{i+1} \). Note that \( T^N \mathcal{G}_{i+1} = \cup_{(I, \rho) \in \mathcal{G}_{i+1}} T^N \mathcal{G}_{i+1} \), where \( \mathcal{G}_{i+1} \) is a singleton \((a_0, \varepsilon_0)\)-standard family containing only \((I, \rho) \). Thinking of \( \mathcal{G}_{i+1} \) (and similarly \( \mathcal{G}_{i+1} \)) as a union of singleton \((a_0, \varepsilon_0)\)-standard families we can apply Lemma 6. However, an intermediate technical step is necessary to properly justify the application of Lemma 6. In the following paragraph we describe this intermediate step.

Suppose \((I, \rho) \) is an element of \( \mathcal{G}_{i+1} \) and it has associated weight \( v \). We replace this element by countably many elements which are the same except that their weights are given by \( \{v v/|\mathcal{G}_{i+1}| \}_{v \in \mathcal{G}_{i+1}} \). Here we have slightly abused notation and labeled these \((a_0, \varepsilon_0)\)-standard pairs by their weights. Similarly, we replace every element in \( \tilde{\mathcal{G}} \) of weight \( \tilde{v} \) by elements of weight \( \{v v/|\mathcal{G}_{i+1}| \}_{v \in \mathcal{G}_{i+1}} \). For every \( v v/|\mathcal{G}_{i+1}| \in \mathcal{G}_{i+1} \), there exists a matching element \( v v/|\mathcal{G}_{i+1}| \in \mathcal{G}_{i+1} \). We apply Lemma 6 to these two elements. As a result, the weight \( v v/|\mathcal{G}_{i+1}| \) is reduced by \( \min\{|v v/|\mathcal{G}_{i+1}|, v v/|\mathcal{G}_{i+1}|\} \gamma \).

Do this for all elements \( v v/|\mathcal{G}_{i+1}| \in \mathcal{G}_{i+1} \). Then the total weight \( |\mathcal{G}_{i+1}| \) is reduced by

\[
\sum v \sum v \min\{|v v/|\mathcal{G}_{i+1}|, v v/|\mathcal{G}_{i+1}|\} \gamma = \sum v \sum v \min\{|v v/|\mathcal{G}_{i+1}|, v v/|\mathcal{G}_{i+1}|\} \gamma.
\]
Observe that this is just \( \min(|\mathcal{G}_L|, |\tilde{\mathcal{G}}_L|) \gamma \). Note that we have just described a matching of weights and nothing else. This intermediate step does not affect any other characteristics of our standard families.

With the above considerations, we obtain a modified \((2a_0, \varepsilon_0)\)-standard family \((\mathcal{G}_L)_N\) such that \( |(\mathcal{G}_L)_N| \leq |\mathcal{G}_L| - \min\{|\mathcal{G}_L|, |\tilde{\mathcal{G}}_L|\} \gamma \). Similarly, \( |(\tilde{\mathcal{G}}_L)_N| \leq |\tilde{\mathcal{G}}_L| - \min\{|\mathcal{G}_L|, |\tilde{\mathcal{G}}_L|\} \gamma \). Let \( \mathcal{G}_N^* \) denote the \((2a_0, \varepsilon_0)\)-standard family consisting of elements of \((\mathcal{G}_L)_N\) and \( T^N \mathcal{G}_S \). Since the standard pairs in \( \mathcal{G}_N^* \) are not modified, we have \( |T^N(\mathcal{G}_S)| = |\mathcal{G}_S| \). Since \( |\mathcal{G}_N^*| = |T^N \mathcal{G}_S| + |(\mathcal{G}_L)_N| \), we have

\[
|\mathcal{G}_N^*| \leq |\mathcal{G}_S| + |\mathcal{G}_L| - \min\{|\mathcal{G}_L|, |\tilde{\mathcal{G}}_L|\} \gamma \\
\leq |\mathcal{G}| - (2/3) \min\{|\mathcal{G}|, |\tilde{\mathcal{G}}|\} \gamma.
\]

Similar estimate is obtained for \( |\tilde{\mathcal{G}}_N^*| \).

\[\square\]

Remark 18 (Recovery of regularity). \( \mathcal{G}_N^* \) and \( \tilde{\mathcal{G}}_N^* \) are \((2a_0, \varepsilon_0)\)-standard families because by Lemma 4, an element \((I_N, \rho_N)\) in one of these families only satisfies \( H(\rho_N) \leq 2a_0 \). Let

\[ n_1 = \lceil (\alpha \ln(\Lambda))^{-1} \ln(1/2 - D/(2a_0)) \rceil. \]

Then applying Lemma 2 we get \( H(\rho_{N+n_1}) \leq 2a_0(\Lambda^{\alpha n_1} + (2a_0)^{-1}D) < a_0 \), for \((I_{N+n_1}, \rho_{N+n_1})\) an element of the \( n_1 \)-th iterate of \((I_N, \rho_N)\). Therefore, \( T^{n_1} \mathcal{G}_N^*, T^{n_1} \tilde{\mathcal{G}}_N^* \) are \((a_0, \varepsilon_0)\)-standard families.

Remark 19 (Recovery of boundary). We also have to worry about the boundary of the standard family after modification. Recall that during modification, we first split a standard pair into two, a constant one and the remainder, then we further split the constant one into at most two pieces. The latter modification also modifies the boundary. However, since the splitting is done on a \( C_X \)-good overlap set, by a crude estimate the splitting increases the total boundary of the family by a factor of \((1 + C_X)\).

Hence \( |\partial \mathcal{G}_N^*| \leq (1 + C_X)|\partial T^N \mathcal{G}| \) and since \( N \geq M \), this is bounded by \( (1 + C_X)|\partial \mathcal{G}| \varepsilon \) for every \( \varepsilon < \varepsilon_0 \). In order to recover from this, we iterate \( \mathcal{G}_N^* \) in multiples of \( n_0 \) and use (5.9). Indeed,

\[
|\partial T^{k\alpha n_0} \mathcal{G}_N^*| \leq B_0|\mathcal{G}_N^*|\varepsilon ((1 + C_X)\partial_1^k + \zeta_2/B_0) \leq B_0|\mathcal{G}|\varepsilon ((1 + C_X)\partial_1^k + \zeta_2/B_0).
\]

To finish the estimate recall our choice of \( B_0 \) and note that we just need to choose \( k = k_0 \in \mathbb{N} \) such that \((1 + C_X)\partial_1^{k_0} + \zeta_2/B_0 < 1 \). Let \( n_2 = k_0n_0 \) and \( n = N + \max\{n_1, n_2\} \).

As a corollary of the above remarks we get the following recovered version of Lemma 7, which can be iterated.

Proposition 2. Suppose \( \mathcal{G} \) and \( \tilde{\mathcal{G}} \) are \((a_0, \varepsilon_0, B_0)\)-proper standard families. There exist \((a_0, \varepsilon_0, B_0)\)-proper standard families \( \mathcal{G}_n^*, \tilde{\mathcal{G}}_n^* \) such that \( \rho_{2^n} = \rho_{2^n \tilde{\mathcal{G}}} = \rho_{T^n \mathcal{G}} = \rho_{T^n \tilde{\mathcal{G}}} \); and,

\[
|\mathcal{G}_n^*| \leq |\mathcal{G}| - \min\{|\mathcal{G}|, |\tilde{\mathcal{G}}|\} \gamma_1,
\]

\[
|\tilde{\mathcal{G}}_n^*| \leq |\tilde{\mathcal{G}}| - \min\{|\mathcal{G}|, |\tilde{\mathcal{G}}|\} \gamma_1.
\]

Remark 20. Note that if \( |\mathcal{G}| = |\tilde{\mathcal{G}}| \) then the right-hand side of the above inequalities is \((1 - \gamma_1)|\mathcal{G}|\).

We are ready to prove our main theorem.
Proof of Theorem 1. Let \( G'_n \) denote the modification of \( T^n G'_n \), where \( G'_n \) is in turn the modification of \( T^n G \). Using Proposition 2 repeatedly, and noting that the weight of the families remain equal before and after modification (i.e. if \( |G| = |\tilde{G}| \), then \( |G'| = |\tilde{G}'| \)), we get

\[
\|L^{2n} \rho - L^{2n} \tilde{\rho}\|_{L^1} \leq |G'_n| + |\tilde{G}'_n| \leq (1 - \gamma_1)|G'_n| + (1 - \gamma_1)|\tilde{G}'_n| \\
\leq (1 - \gamma_1)^2 |G| + (1 - \gamma_1)^2 |\tilde{G}| \\
\leq 2(1 - \gamma_1)^2.
\]

For a general \( m \in \mathbb{N} \), write \( m = k\tilde{n} + r \), where \( 0 \leq r < \tilde{n} \). Using \( \|L^r \rho\|_{L^1} \leq \|\rho\|_{L^1} \), we obtain \( \|L^m \rho - L^m \tilde{\rho}\|_{L^1} \leq \|L^{kn} \rho - L^{kn} \tilde{\rho}\|_{L^1} \). This is bounded by \( 2(1 - \gamma_1)^k \leq 2(1 - \gamma_1)^{(m/\tilde{n})-1} = 2(1 - \gamma_1)^{-1} ((1 - \gamma_1)^{1/\tilde{n}})^m \).

A simple consequence of Proposition 2 is that for every \((a_0, \varepsilon_0, B_0)\)-proper standard pair \((I, \rho)\), the sequence \(\{L^m \rho\}_m\) is a Cauchy sequence in \(L^1(X, \mathcal{B}, m)\) hence it has a limit \(\varphi \in L^1\). Moreover, this limit does not depend on the choice of the starting standard pair \(\rho\). Indeed, for \(n > m\), \(\|L^m \rho - L^m \varphi\|_{L^1} = \|\rho - \varphi\|_{L^1}\), where \(G = \{(I, \rho)\}\) and \(\tilde{G} = T^n m \mathcal{G}\), which are \((a_0, \varepsilon_0, B_0)\)-proper probability standard families. Applying Proposition 2 repeatedly (as in the proof of Theorem 1), shows that \(\|\rho - \varphi\|_{L^1}\) can be made arbitrarily small if \(n\) and \(m\) are sufficiently large and the result follows.

Remark 21. The notion of being an \((a_0, \varepsilon_0, B_0)\)-proper standard family is a notion of regularity. Let us briefly comment on its relation to the notion of Hölder regularity. More precisely we will show that certain Hölder functions can be represented as \((a_0, \varepsilon_0, B_0)\)-proper standard families. Therefore, two such functions converge exponentially to one another under iteration.

We say that \(V \subset X\) is \((a_0, \varepsilon_0, B_0)\)-nice if there exists a (mod 0)-partition \(\{V_t\}\) of \(V\) into countably many open sets such that diam \(V_t \leq \varepsilon_0\), \(\forall t\), and \(m(\partial V_t) \leq e^{-a_0\varepsilon_0} B_0 m(V_t), \forall \varepsilon \leq \varepsilon_0\).

Suppose \(f \in C^0(X, \mathbb{R})\) is a bounded, Hölder continuous function supported on a \((a_0, \varepsilon_0, B_0)\)-nice set \(V \subset X\) with \(m(V) < \infty\) and \(a_0 > 0\). Choosing \(c = |f|_{a_0} + \sup |f|\), we can write \(f = f + c - c\), where \(H(f + c) \leq a_0\). It is easy to see that \(f + c\) can be written as an \((a_0, \varepsilon_0, B_0)\)-proper standard family. Indeed, the normalized restrictions of \(f + c\) to sets \(V_t\) form an \((a_0, \varepsilon_0, B_0)\)-standard family \(\mathcal{G}\) and \(\|\partial_v \mathcal{G}\| \leq e^{c/m} \sum m(\partial V_t) \leq B_0 \varepsilon m(V)\).

Suppose \(f, g \in C^0(X, \mathbb{R})\) are bounded, Hölder continuous functions supported on \((a_0, \varepsilon_0, B_0)\)-nice sets \(V_f, V_g\) such that \(m(V_f) = m(V_g) < \infty\) and \(\int_{V_f} f = \int_{V_g} g\). Write \(f = f + c - c\) and \(g = g + c - c\), where \(c = \max \{\sup |f|, \sup |g|\}/a_0 + \max \{\sup |f|, \sup |g|\}\). Suppose our dynamical system satisfies conditions (1)-(5) with parameters \(a_0, \varepsilon_0, B_0\). Then, applying Proposition 2,

\[
\|L^m f - L^m g\|_{L^1} \leq \|L^m (f + c) - L^m (g + c)\|_{L^1} \\
\leq \|\rho - \varphi\|_{L^1} \leq 2C\gamma_1^{m} |\mathcal{G}|,
\]

where \(|\mathcal{G}| = \int_{V_f} (f + c) \leq \frac{2m(V_f)}{a_0} \max \{\|f\|_{L^1}, \|g\|_{L^1}\}\).

7. Inducing schemes

Throughout this section we assume conditions (1)-(4) of Section 2 hold. Condition (5) is not needed, but it is replaced by an additional assumption, namely (6), (7) or (8) below. All of these assumptions ask for the existence of a (mod 0)-partition of the space \(X\) which is then used to build a suitable inducing scheme.
The content of this section is independent of Section 6.

7.1. Inducing scheme 1. In this subsection we assume the following.

(6) Partition \( \mathcal{R} \): There exist a finite (mod 0)-partition \( \mathcal{R} = \{ R_j \}_{j=1}^{N} \) of \( X \) into open sets (recall Definition 2) such that

1. for every \( 1 \leq j \leq N \), \( \sup_{\varepsilon > 0} \varepsilon^{-1} m(\partial \varepsilon R_j) < \infty \),
2. \( \exists c_\mathcal{R} \in (0, 1), C_\mathcal{R} > 0 \), possibly depending on \( \delta_0 \), s.t. for every \( \delta_0 \)-regular set \( I \), \( \text{diam } I \leq \varepsilon_0 \), there exists \( R = R(I) \in \mathcal{R} \) s.t. \( I \supset R \) and

\[
\text{if } m(I \setminus R) \neq 0, \text{ then } m(I \setminus R) \geq c_\mathcal{R} m(I); \quad (7.1)
\]

\[
m(\partial \varepsilon (I \setminus \text{cl } R) \setminus \partial \varepsilon I) \leq C_\mathcal{R} m(\partial \varepsilon I). \quad (7.2)
\]

Remark 22. Item (1) implies that \( m(\partial R) = 0, \forall R \in \mathcal{R} \). It follows that \( m(I \setminus \text{cl } R) = m(I \setminus R) \).

Under assumptions (1)-(4), (6) we construct an inducing scheme where the base map is a Gibbs-Markov map with finitely many images and the return times have exponential tails.

Proposition 3. There exists a refinement \( \mathcal{P}' \) of the partition \( \mathcal{P} = \{ O_R \} \) for \( T : X \ominus \) into open sets (mod 0) and a map \( \tau : X \to \mathbb{Z}^+ \) constant on elements of \( \mathcal{P}' \) such that

1. The map \( G = T^\tau : X \ominus \) is a Gibbs-Markov map with finitely many images \( \{ Z_1, Z_2, \ldots, Z_q \} \subset \mathcal{R} \).
2. \( m(\tau > n) \leq \text{const} \cdot \kappa^n \) for some \( \kappa \in (0, 1) \).

Before we prove Proposition 3, we need several lemmas.

Lemma 8 (Remainder family \( \hat{\mathcal{G}} \)). Suppose \( \mathcal{G} \) is an \( (a_0, \varepsilon_0, B_0) \)-proper standard family. Let \( \hat{\mathcal{G}} \) be the family obtained from \( \mathcal{G} \) by replacing each \( (I_j, \rho_j) \) of weight \( w \), having a \( \delta_0 \)-regular domain and containing an element \( R = R(I) \in \mathcal{R} \) in its domain with \( m(I \setminus R) \neq 0 \), by \( (I \setminus \text{cl } R, \rho I \setminus \text{cl } R / \int_{I \setminus \text{cl } R} \rho) \) of weight \( w \int_{I \setminus \text{cl } R} \rho \). Then \( \hat{\mathcal{G}} \) is an \( (a_0, \varepsilon_0, C_R B_0) \)-proper standard family, where \( C_R = (e^{a_0 \varepsilon_0^2} C_R + 1) e^{a_0 \varepsilon_0^2} C_R^{-1} \).

Proof. This is a consequence of item (2) of (6). Indeed, assuming \( \mathcal{G} = \{(I_j, \rho_j)\} \) with associated weights \( w_j \), we have, \( \forall \varepsilon < \varepsilon_0 \),

\[
|\partial \varepsilon \hat{\mathcal{G}}| \leq \sum_j w_j \int_{\partial \varepsilon I_j \setminus \text{cl } R} \rho_j \leq \sum_j w_j \left( \int_{\partial \varepsilon (I_j \setminus \text{cl } R) \setminus \partial \varepsilon I_j} \rho_j + \int_{\partial \varepsilon I_j} \rho_j \right) \leq \sum_j \left( e^{a_0 \varepsilon_0^2} \frac{m(\partial \varepsilon (I_j \setminus \text{cl } R) \setminus \partial \varepsilon I_j)}{m(\partial \varepsilon I_j)} \int_{\partial \varepsilon I_j} \rho_j + \int_{\partial \varepsilon I_j} \rho_j \right) \leq \left( e^{a_0 \varepsilon_0^2} C_R + 1 \right)|\partial \varepsilon \mathcal{G}|,
\]

where in the second line we have used the Comparability Lemma 1 and in the last line we have used (7.2). Since \( \mathcal{G} \) is \( B_0 \)-proper, \( |\partial \varepsilon \mathcal{G}| \leq B_0 \varepsilon |\mathcal{G}| \); moreover (7.1) can be used to show that \( |\hat{\mathcal{G}}| \leq e^{a_0 \varepsilon_0^2} C_R^{-1} |\hat{\mathcal{G}}| \). Indeed, by Lemma 1,

\[
|\hat{\mathcal{G}}| \geq \sum_j w_j \int_{I_j \setminus \text{cl } R} \rho_j \geq \sum_j w_j e^{-a_0 \varepsilon_0^2} \frac{m(I_j \setminus R)}{m(I_j)} \int_{I_j} \rho_j \geq e^{-a_0 \varepsilon_0^2} C_R |\mathcal{G}|.
\]

It follows that \( |\partial \varepsilon \hat{\mathcal{G}}| \leq C_R B_0 \varepsilon |\hat{\mathcal{G}}| \). \( \square \)

Lemma 9. Let \( \mathcal{R} = \{ R_k \}_{k=1}^{N} \) be the partition from (6). There exists a constant \( t > 0 \) such that if \( \mathcal{G} = \{(I_j, \rho_j)\}_{j \in J} \) is an \( (a_0, \varepsilon_0, B_0) \)-proper standard family, then
The following steps lead to our sought after inducing scheme.

Proof of Proposition 3.

where \( J_{\text{reg}} \) is the set of \( j \in J \) such that \( I_j \) is \( \delta_0 \)-regular

Proof.

Since \( G \) is an \((a_0, \varepsilon_0, B_0)\)-proper standard family, at least 2/3 of its weight is concentrated on \((a_0, \varepsilon_0)\)-standard pairs \((I, \rho)\), where \( I \) is a \( \delta_0 \)-regular set (recall that \( \delta_0 = 1/(3B_0) \)). By item (2) of (6), each such standard pair contains an element from the collection \( R \). Using this fact and the regularity of standard pairs (recall (5.1)), the left-hand side of (7.3) is

\[
\sum_{j \in J_{\text{reg}}} w_j \int_{R(I_j)} \rho_j \geq t \cdot \left( \sum_{j \notin J_{\text{reg}}} w_j + \sum_{j \in J_{\text{reg}}} w_j \int_{I_j \backslash R(I_j)} \rho_j \right), \tag{7.3}
\]

Now consider the expression in the parentheses and on the right-hand side of (7.3). The first term of this expression is the total weight of the standard pairs that are not \( \delta_0 \)-regular so this term is \( \leq (1/3)\|G\| \). The second term represents the weights of the remainders, after removing \( c \ell R(I_j) \), from each \( \delta_0 \)-regular \( I_j \). This sum is

\[
geq (2/3)\|G\| e^{-a_0 \varepsilon_0^2} \frac{m(R(I_j))}{m(I_j)} \geq (2/3) e^{-a_0 \varepsilon_0^2} C_{\text{ball}}(\varepsilon_0)^{-1} m(R(I_j)) \]

where \( m(R) = \min_{1 \leq k \leq N} m(R_k) \). So the expression in the parentheses and on the right-hand side of (7.3) is \( \leq |G| (1/3 + e^{-a_0 \varepsilon_0^2} C_{\text{ball}}(\varepsilon_0)/m(R)) \). Therefore the inequality (7.3) is satisfied if we take:

\[
t = \frac{(2/3)\|G\| e^{-a_0 \varepsilon_0^2} C_{\text{ball}}(\varepsilon_0)^{-1} m(R)}{|G|(1/3 + e^{-a_0 \varepsilon_0^2} C_{\text{ball}}(\varepsilon_0)/m(R))} = \frac{(2/3)e^{-a_0 \varepsilon_0^2} C_{\text{ball}}(\varepsilon_0)^{-1} m(R)^2}{(1/3)m(R) + e^{-a_0 \varepsilon_0^2} C_{\text{ball}}(\varepsilon_0)}. \tag{7.4}
\]

Proof of Proposition 3. The following steps lead to our sought after inducing scheme.

(1) Consider the partition \( \mathcal{R} \) of \( X \). Let us focus on defining the inducing scheme on one element of this partition. The same can be done for all other partition elements and in a uniform way because \( \mathcal{R} \) is finite. Fix \( R \in \mathcal{R} \) and let \( G_0 = \{ \{ R, \mathbf{1}_R/m(R) \} \} \) and \( w_0 = m(R) > 0 \). Due to item (1) of (6), the singleton family \( G_0 \) with associated weight \( \{ w_0 \} \) is an \((a_0, \varepsilon_0, B)\)-proper standard family \( G_0 \) for some constant \( B > 0 \) possibly larger than \( B_0 \).

(2) By Remark 17, \( G_1 := T_{n_{\text{rec}}(B)} G_0 \) is an \((a_0, \varepsilon_0, B_0)\)-proper standard family.

(3) By item (2) of (6), every standard pair in \( G_1 \) whose domain is \( \delta_0 \)-regular contains an element \( R_k \), \( 1 \leq k \leq N \), from the collection \( \mathcal{R} \). “Stop” such standard pairs of \( G_1 \) on \( R_k \in \mathcal{R} \). By stopping we mean going back to \( R \) and defining the return time \( \tau = n_{\text{rec}}(B) \) on the subset of \( R \) that maps onto \( R_k \) under \( T_{n_{\text{rec}}(B)} \). By Lemma 9, the ratio of the removed weight from \( G_1 \) to the weight of the remainder family, which we denote by \( G_0' \), is at least some positive constant \( t \) given by (7.4). Note that since total weight is preserved during iteration, this corresponds to defining \( \tau \) on a subset \( A \subset R \) such that \( m(A) \geq t \cdot m(R \setminus A) \). Also, by Lemma 8, \( G_1 \) is an \((a_0, \varepsilon_0, C \mathcal{R} B_0)\)-standard family.

(4) Just as in step (2), \( G_2 := T_{n_{\text{rec}}(C \mathcal{R} B_0)} G_1 \) is an \((a_0, \varepsilon_0, B_0)\)-proper standard family so we can apply step (3) to it.

(5) Repeat the steps (3), (4) \( \rightarrow \) (3), (4) \( \rightarrow \) \( \cdots \), incrementing the indices accordingly during the process.
Proposition 4. There exists a refinement $\tilde{G}$ of the partition $\mathcal{P}$ and $\tau$ is constant on each element of $\mathcal{P}$.

Lemma 8 that concerns the set $m$ and satisfying $m(A) \geq t \cdot m(X \setminus A)$. By construction the induced map has finitely many images which form a sub-collection of $\mathcal{R}$. Note that distortion bound is always maintained under iterations of $T$ by assumptions (1) and (2) so we need not worry about it.

7.2. Inducing scheme 2. In this subsection we make a stronger assumption than (6), but also prove a stronger result in which the inducing scheme has a full-branched Gibbs-Markov base map and a return time that has $\gcd = 1$ in addition to having exponential tails. Such an inducing scheme is much more useful in obtaining statistical properties of $T$ beyond the existence of finitely many ACIPs.

We assume the following.

(7) Partition $\mathcal{R}$: There exist a finite (mod 0)-partition $\mathcal{R} = \{R_j\}_{j=1}^N$ of $X$ into open sets such that

1. for every $1 \leq j \leq N$, $\sup_{\varepsilon > 0} \varepsilon^{-1} m(\partial R_j) < \infty$,
2. $\exists \varepsilon \in (0, 1), C_\varepsilon > 0$, possibly depending on $\delta_0$, s.t. for every $\delta_0$-regular set $I$, $\text{diam} I \leq \varepsilon_0$, there exists $R = R(I) \in \mathcal{R}$ s.t. $I \supset R$ and
   \[
   \text{if } m(I \setminus R) \neq 0, \text{ then } m(I \setminus R) \geq c_\varepsilon m(I); \quad (7.5)
   \]
   \[
   m(\partial_\varepsilon(I \setminus \text{cl} R) \setminus \partial_\varepsilon I) \leq C_\varepsilon m(\partial_\varepsilon I). \quad (7.6)
   \]
3. $\exists Z \in \mathcal{R}$ s.t.
   \[
   \text{diam } Z \leq \eta \varepsilon_0; \quad (7.7)
   \]
   for every $\delta_0$-regular set $I \subset X$, $\text{diam } I \leq \varepsilon_0$ and $m(I \setminus Z) \neq 0$,
   \[
   m(I \setminus Z) \geq c_\varepsilon m(I), \quad (7.8)
   \]
   \[
   m(\partial_\varepsilon(I \setminus \text{cl} Z) \setminus \partial_\varepsilon I) \leq C_\varepsilon m(\partial_\varepsilon I) \quad (7.9)
   \]
   \[
   \gcd \{n : T^n Z \supset Z\} = 1. \quad (7.10)
   \]

Remark 23. Notice that the first two items of (7) are the same as (6).

Under assumptions (1)-(4), (7) we prove the following.

Proposition 4. There exists a refinement $\mathcal{P}'$ of the partition $\mathcal{P}$ of $T : X \supset$ into open sets (mod 0), a set $Z$ (the one from (7)) consisting of elements of $\mathcal{P}'$ and a map $\tilde{\tau} : Z \to \mathbb{Z}^+$ constant on elements of $\mathcal{P}'$ such that

(a) The map $\tilde{G} = T^{\tilde{\tau}} : Z \supset$ is a full-branched Gibbs-Markov map.
(b) $\gcd \{n \geq 1 : m(\tilde{\tau} = n) > 0\} = 1$.
(c) $\text{m}(\tilde{\tau} > n) \leq \text{const} \cdot \kappa^n$ for some $\kappa \in (0, 1)$.

Before we get to the proof of this proposition we need a slight variation of Lemma 8 that concerns the set $Z$. Setting $R = Z$, the only difference to Lemma 8 is that we do not require $I \supset R$. The proof is essentially the same as the proof of Lemma 8; nevertheless, we provide it.

Lemma 10. Suppose $\tilde{G}$ is an $(a_0, \varepsilon_0, B_0)$-proper standard family. Let $\hat{G}$ be the family obtained from $G$ by replacing each $(I, \rho)$ of weight $w$, having a $\delta_0$-regular domain, and satisfying $m(I \setminus Z) \neq 0$, by $(I \setminus \text{cl} Z, \rho I \setminus \text{cl} Z)$ of weight $w I \setminus Z$. Then $\hat{G}$ is an $(a_0, \varepsilon_0, C_\varepsilon B_0)$-proper standard family, where $C_\varepsilon = (e^{\varepsilon_0} C_\varepsilon + 1)e^{\varepsilon_0} c_\varepsilon^{-1}$. 

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Proof. This is a consequence of (7.9). Indeed, assuming \( \mathcal{G} = \{(I_j, \rho_j)\} \) with associated weights \( w_j \), we have, \( \forall \varepsilon < \varepsilon_0 \),

\[
|\partial_\varepsilon \mathcal{G}| \leq \sum_j w_j \int_{\partial_c(I_j \setminus cl Z)} \rho_j \leq \sum_j w_j \left( \int_{\partial_c(I_j \setminus cl Z) \setminus \partial_c I_j} \rho_j + \int_{\partial_c I_j} \rho_j \right)
\]

\[
\leq \sum_j w_j \left( e^{a_0 \varepsilon} \frac{m(\partial_c I_j \setminus cl Z) \setminus \partial_c I_j)}{m(\partial_c I_j)} \int_{\partial_c I_j} \rho_j + \int_{\partial_c I_j} \rho_j \right)
\]

\[
\leq (e^{a_0 \varepsilon} C_R + 1)|\partial_\varepsilon \mathcal{G}|,
\]

where in the second line we have used the Comparability Lemma 1 and in the last line we have used (7.9). Since \( \mathcal{G} \) is \( B_0 \)-proper, \( |\partial_\varepsilon \mathcal{G}| \leq B_0|\mathcal{G}| \); moreover (7.8) can be used to show that \( |\mathcal{G}| \leq e^{a_0 \varepsilon} C_R \mathcal{G} \). It follows that \( |\partial_\varepsilon \mathcal{G}| \leq C_R B_0|\mathcal{G}| \). \( \square \)

Proof of Proposition 4. Let \( Z \subseteq X \) be as in (7). Let \( \mathcal{N}_Z = \{ n : T^n Z \supset Z \} \). Since \( \gcd(N_Z) = 1 \), there exists \( K \in \mathbb{N} \) and \( \tilde{n}_j \}_{j=1}^K \subset \mathcal{N}_Z \) such that \( \gcd(\tilde{n}_j) \}_{j=1}^K = 1 \). Let us assume that \( \tilde{n}_1 < \tilde{n}_2 < \cdots < \tilde{n}_K \). Note that if \( K = 1 \), then \( 1 \in \mathcal{N}_Z \) and therefore \( \mathbb{N} \subset \mathcal{N}_Z \). So without loss of generality we can assume that \( K \geq 2 \).

Now we follow a line of reasoning similar to that of the proof of Proposition 3, but with some modifications when dealing with \( R = \mathbb{Z} \) mainly in order to achieve item (b) of Proposition 4.

1. Let \( \mathcal{G}_0 = \{ Z, \mathbb{1}_Z / m(Z) \} \) and \( w_0 = \mathbb{1}_m(Z) \). \( \mathcal{G}_0 \) is an \((a_0, \varepsilon_0, B)\)-proper standard family for some \( B > 0 \).

   Let \( m_1, m_2 \in \mathbb{N} \) be s.t. \( \tilde{n}_1 + m_1 \tilde{n}_K \geq n_{rec}(B) \) and \( m_2 \tilde{n}_K \geq n_{rec}(C_R B_0) \).

   \[
   n_0 = \max\{m_1, m_2\} \text{ and define } \{n_j\}_{j=1}^K \text{ by }
   \]

   \[
   n_j := \tilde{n}_j + m_0 \tilde{n}_K, \text{ if } 1 \leq j \leq K - 1;
   \]

   \[
   n_K := \tilde{n}_K + \sum_{j=1}^{K-1} n_j.
   \]

   It is a simple exercise to verify that \( n_1 < n_2 < \cdots < n_K \), \( \gcd(\{n_j\}_{j=1}^K) = 1 \) and \( \{n_j\}_{j=1}^K \subset \mathcal{N}_Z \). The benefit of \( \{n_j\}_{j=1}^K \) over \( \{\tilde{n}_j\}_{j=1}^K \) is that \( n_1 \geq n_{rec}(B) \) and \( n_{j+1} - n_j \geq n_{rec}(C_R B_0), \forall j \in \{1, \ldots, K - 1\} \).

2. Let \( \mathcal{G}_1 := T^{n_1} \mathcal{G}_0 \), taking \( V_\varepsilon = cl Z \) as the set to avoid under \( T^{n_1} \) under artificial chopping. This can be done due to (7.7) and condition (4) on divisibility of large sets. Since \( n_1 \geq n_{rec}(B) \), \( \mathcal{G}_1 \) is an \((a_0, \varepsilon_0, B_0)\)-proper standard family.

   (2.1) Define \( \tau = n_1 \) on \( A_1 := T^{-n_1} Z \cap Z \). Note that \( T^{n_1}(Z) \supset Z \). By Lemma 10, the remainder from \( \mathcal{G}_1 \), which we denote by \( \mathcal{G}_1 \), is an \((a_0, \varepsilon_0, C_R B_0)\)-proper standard family. Let \( \mathcal{G}_2 = T^{n_2-n_1} \mathcal{G}_1 \). Since \( n_2 - n_1 \geq n_{rec}(C_R B_0) \), \( \mathcal{G}_2 \) is an \((a_0, \varepsilon_0, B_0)\)-proper standard family.

   (2.2) Define \( \tau = n_2 \) on \( A_2 := T^{-n_2} Z \cap (Z \setminus A_1) \). By Lemma 10, the remainder from \( \mathcal{G}_2 \), which we denote by \( \mathcal{G}_2 \), is an \((a_0, \varepsilon_0, C_R B_0)\)-proper standard family. Let \( \mathcal{G}_3 = T^{n_3-n_2} \mathcal{G}_2 \). Since \( n_3 - n_2 \geq n_{rec}(C_R B_0) \), \( \mathcal{G}_3 \) is an \((a_0, \varepsilon_0, B_0)\)-proper standard family.

   (2.3) We continue this process until we define \( \tau = n_K \) on

\[
A_K := T^{-n_K} Z \cap (Z \setminus \bigcup_{j=1}^{K-1} A_j).
\]
Let \( \hat{G}_K \) be the remainder from \( G_K \). Note that \( \hat{G}_K \) is an \((a_0, \varepsilon_0, C_\mathcal{R} B_0)\)-proper standard family. Also note that \( \forall j \in \{1, \ldots, K\}, A_j \subset \hat{Z} \) and \( T^{n_j} A_j = \hat{Z} \). Moreover, \( m(A_j) > 0 \) because \( \forall j \in \{1, \ldots, K\} \) the inverse branches of \( T^{n_j} \) are non-singular, there are at most countably many such branches and \( m(\hat{Z}) > 0 \).

3. We have achieved that

\[
\gcd \left\{ n : \mathbf{m} \left( \{ \tau = n \} \cap \bigcup_{j=1}^{K} A_j \right) > 0 \right\} = 1.
\]

We continue the construction of \( \tau \) on the rest of \( Z \), i.e. on \( \hat{Z} = Z \setminus \bigcup_{j=1}^{K} A_j \), in such a way that it has exponential tails. We will do so by continuing to iterate \( \hat{G}_K \).

4. Let \( \hat{G}_{K+1} = T^{n_{\text{rec}}(C_\mathcal{R} B_0)} \hat{G}_K \). Then \( \hat{G}_{K+1} \) is an \((a_0, \varepsilon_0, B_0)\)-proper standard family. By item (2) of (7), every standard pair in \( \hat{G}_{K+1} \) whose domain is \( \delta_0\)-regular contains an element \( R_k \), \( 1 \leq k \leq N \), from the collection \( \mathcal{R} \).

“Stop” such standard pairs of \( \hat{G}_{K+1} \) on \( R_k \in \mathcal{R} \). By stopping we mean going back to \( \hat{Z} \subset Z \) and defining the return time \( \tau = n_K + n_{\text{rec}}(C_\mathcal{R} B_0) \) on the subset of \( \hat{Z} \) that maps onto \( R_k \) under \( T^{n_K + n_{\text{rec}}(C_\mathcal{R} B_0)} \). By Lemma 9, the ratio of the removed weight from \( \hat{G}_{K+1} \) to the weight of the remainder family, which we denote by \( \hat{G}_{K+1} \), is at least some positive constant \( t \) given by (7.4). Note that since the total weight is preserved under iteration, this corresponds to defining \( \tau \) on a subset \( A \subset \hat{Z} \) such that \( m(A) \geq t \cdot m(\hat{Z} \setminus A) \).

Also, by Lemma 8, \( \hat{G}_{K+1} \) is an \((a_0, \varepsilon_0, C_\mathcal{R} B_0)\)-standard family.

5. \( \hat{G}_{K+2} := T^{n_{\text{rec}}(C_\mathcal{R} B_0)} \hat{G}_{K+1} \) is an \((a_0, \varepsilon_0, B_0)\)-proper standard family so we can apply step (4) to it.

6. Repeat the steps (4), (5) \( \rightarrow \) (4), (5) \( \rightarrow \) \( \ldots \), incrementing the indices accordingly during the process. This procedure defines \( \tau \) on \( \hat{Z} \) up to a measure zero set of points (which includes points that map into \( \partial \hat{Z} \)).

The above steps described how to define \( \tau \) on \( Z \). We have also explained how to define \( \tau \) on the rest of the elements of \( \mathcal{R} \) in Section 7.1 (Recall that condition (7) is stronger than (6) so the results of the previous subsection are valid). Putting these together we get the same statement as Proposition 3, but with the additional properties that \( \gcd \{ n : \mathbf{m} \{ \tau = n \} > 0 \} = 1, Z \) is one of the finitely many images of \( G = T^\tau \) and that \( G(Z) \supset Z \).

Let \( \varsigma : Z \rightarrow \mathbb{N} \) be the first return time of \( G \) to \( Z \) and \( \hat{G} = G^\varsigma : Z \supset \) be the associated first return map. Since \( G A_j = T^\tau A_j = T^{n_j} A_j = Z, \forall j \in \{1, \ldots, K\} \), it follows that \( \varsigma = 1 \) on the set \( \bigcup_{j=1}^{K} A_j \).

Define \( \bar{\tau} = \sum_{j=0}^{\varsigma-1} \tau \circ G^j : Z \rightarrow \mathbb{N} \), then \( \hat{G} = T^{\bar{\tau}} \). It follows from the previous paragraph that \( \bar{\tau} = \tau \) on \( \bigcup_{j=1}^{K} A_j \subset \hat{Z} \). This implies item (b). Item (a) and item (c) simply follow from the fact that \( G \) is a Markov map with finitely many states (hence \( \varsigma \) has exponential tails) and \( \tau : X \rightarrow \mathbb{N} \) has exponential tails.

\( \square \)

7.3. Inducing scheme 3. In this subsection we make an assumption which is again stronger than (6), but different from (7). The assumption contains more dynamical information than (7) and leads to an improvement of Item (b) of Proposition 4. The advantage of this improvement is that it makes it easier to connect this inducing scheme to other inducing schemes, say if one is interested in a system that initially is not piecewise expanding but admits (somehow) an inducing scheme with a base map that is piecewise expanding.

We assume the following.
(8) **Partition** $\mathcal{R}$: There exist a finite (mod 0)-partition $\mathcal{R} = \{R_j\}_{j=1}^N$ of $X$ into open sets such that

1. for every $1 \leq j \leq N$, $\sup_{x>0} e^{-1}m(\partial x R_j) < \infty$, 
2. $\exists C_R \in (0, 1), C_R > 0$, possibly depending on $\delta_0$, s.t. for every $\delta_0$-regular set $I$, $\text{diam } I \leq \xi_0$, there exists $R = R(I) \in \mathcal{R}$ s.t. $I \supset R$ and
   \begin{align}
   \text{if } m(I \setminus R) \neq 0, \text{ then } m(I \setminus R) \geq c_R m(I); \quad (7.11) \\
   m(\partial \tau (I \setminus \text{cl } R) \setminus \partial \tau I) \leq C_R \mu (\partial \tau I). \quad (7.12)
   \end{align}
3. $\exists Z \in \mathcal{R}$ and $Z' \supset Z$ s.t.
   \begin{align}
   \text{diam } Z' \leq \epsilon \epsilon_0; \quad (7.13) \\
   m(Z' \setminus Z) \geq c_R m(Z'); \quad (7.14)
   \end{align}
   and for every open set $I$ with $\text{diam } I \leq \epsilon_0$ and $I \supset Z'$,
   \begin{align}
   m(\partial \tau (I \setminus \text{cl } Z) \setminus \partial \tau I) \leq C_R \mu (\partial \tau I). \quad (7.15)
   \end{align}
   Moreover, there exists a finite collection of partition elements $\mathcal{P}_Z = \{O_k\}_{k=1}^K$ s.t. $\forall k \in \{1, \ldots, K\}$, $O_k \in \mathcal{P}$, $O_k \subset Z$, $TO_k \supset Z'$.

Under assumptions (1)-(4), (8) we prove the following.

**Proposition 5.** There exists a refinement $\mathcal{P}'$ of the partition $\mathcal{P}$ of $T : X \supset$ into open sets (mod 0), a set $Z$ (the one from (8)) consisting of elements of $\mathcal{P}'$ and a map $\tilde{T} : Z \to \mathbb{Z}^+$ constant on elements of $\mathcal{P}'$ such that

(a) The map $\tilde{T} = T \tilde{T} : Z \supset$ is a full-branched Gibbs-Markov map.
(b) $m(\{\tilde{T} = 1\} \cap O_k) > 0$ for every $1 \leq k \leq K$.
(c) $m(\tilde{T} > 1) \leq C \cdot \tilde{k}$ for some $\tilde{k} \in (0, 1)$.

**Proof.** Let $Z \subset X$ be as in (8).

1. Let $\mathcal{G}_0 = \{Z, \mathbb{I}_Z/m(Z)\}$ and $w_0 = m(Z)$. $\mathcal{G}_0$ is an $(a_0, \epsilon_0, B)$-proper standard family for some $B > 0$.
2. Let $\mathcal{G}_1 := T T_{\mathcal{G}_0}$, taking $V_\epsilon = \text{cl } Z' \setminus Z$ as the set to avoid under artificial chopping. This can be done due to (7.13) and condition (4) on divisibility of large sets.
   Define $\tau = 1$ on $A_k := T^{-1} Z \cap O_k$, $\forall k \in \{1, \ldots, K\}$. Note that $TA_k = Z$ since $T(O_k) \supset Z$. Let $\mathcal{G}_1$ be the family obtained from $\mathcal{G}_1$ by replacing each $(1, \rho)$ of weight $w$, containing $Z'$ in its domain by $(1 \setminus \text{cl } Z, \rho \mathbb{I}_{x \in Z} / (1 \setminus \text{cl } Z)$ of weight $w \int_{1 \setminus Z} \rho$. Then it can be shown, following the ideas of the proof of Lemma 8 and item (3) of (8), that $\mathcal{G}_1$ is an $(a_0, \epsilon_0, B')$-proper standard family for some $B' > 0$. Now we proceed as before. Let
   \[ \tilde{Z} := Z \setminus \bigcup_{k=1}^K A_k. \]
3. Let $\mathcal{G}_2 = T^{n_{\text{rec}}(B')}, \mathcal{G}_1$. Then $\mathcal{G}_2$ is an $(a_0, \epsilon_0, B_0)$-proper standard family. By item (2) of (8), every standard pair in $\mathcal{G}_2$ whose domain is $\delta_0$-regular contains an element $R_k$, $1 \leq k \leq N$, from the collection $\mathcal{R}$. “Stop” such standard pairs of $\mathcal{G}_2$ on $R_k \in \mathcal{R}$. By stopping we mean going back to $Z$ and defining the return time $\tau = 1 + n_{\text{rec}}(B')$ on the subset of $\tilde{Z}$ that maps onto $R_k$ under $T^n_{1 + n_{\text{rec}}(B')}$. By Lemma 9, the ratio of the removed weight from $\mathcal{G}_2$ to the weight of the remainder family (remainder as in Lemma 8), which we denote by $\mathcal{G}_2$, is at least some positive constant $t$ given by (7.4). Note that since the total weight is preserved under iteration, this corresponds to defining $\tau$ on a subset $A \subset \tilde{Z}$ such that $m(A) \geq \epsilon \cdot m(\tilde{Z} \setminus A)$. Also, by Lemma 8, $\mathcal{G}_2$ is an $(a_0, \epsilon_0, C_R B_0)$-standard family.
(4) $G_3 := T^{n_{rec}(G_k B_0)}G_2$ is an $(a_0, \varepsilon_0, B_0)$-proper standard family so we can apply step (3) to it.
(5) Repeat the steps (3), (4) → (3), (4) → · · · , incrementing the indices accordingly during the process. This procedure defines $\tau$ on $\hat{Z}$ up to a measure zero set of points.

The above steps described how to define $\tau$ on $\hat{Z}$ so that $\mathfrak{m}(\{\tau = 1\} \cap O_k) > 0$. We have also explained how to define $\tau$ on the rest of the elements of $\mathcal{R}$ in Section 7.1 (Recall that condition (8) is stronger than (6)). The rest of the proof is the same as the proof of Proposition 4.

In the remaining sections we provide specific examples and justify that our assumptions can be checked for various dynamical systems.

8. Example 1: A nonlinear W-map

The following “W-map” example\textsuperscript{7} is taken from [21, Example 2]. The map $T : X = (0, 1) \circlearrowleft$ is given by
\[ T = \begin{cases} T_1 := 1 - (40/9)x, & 0 \leq x < 9/40, \\ T_2 := 2(x - 9/40), & 9/40 \leq x < 9/20, \\ T_3 := -4(x - 9/16), & 9/20 \leq x < 9/16, \\ T_4 := x^2 + (81/128)x - 81/128, & 9/16 \leq x < 1. \end{cases} \]

The graph of this map is depicted in Figure 1.

In [21] explicit estimates on the rate of decay of correlations were obtained for this map using Hilbert metric contraction as in [27]. We use the same example in order to make it possible to compare the explicit values obtained in [21] to those obtained by our method. As we will see, in this case our constants of mixing will be much worse than those in [21] and we will explain the reason at the end of this section.

In this example, $X = (0, 1)$ with its usual metric $d(x, y) = |x - y|$ and Lebesgue measure $\mathfrak{m}$. Also one can take $\varepsilon_1 = 1$. Let us denote by $\mathcal{H} = \{h_j\}_{j=1}^4$ the inverse branches of the map $T$ from left to right. We have $O_{h_1} = (0, 9/40)$, $O_{h_2} = (9/40, 9/20)$, $O_{h_3} = (9/20, 9/16)$, $O_{h_4} = (9/16, 1)$.

In the following subsections we check conditions (1)-(5).

8.1. Uniform expansion. It is easy to check that with $\varepsilon_2 = 1$, the following bounds on the contraction factors hold for the inverse branches of the map $T$.

$\Lambda_{h_1} = 9/40, \Lambda_{h_2} = 1/2, \Lambda_{h_3} = 1/4, \Lambda_{h_4} = 112/207, \Lambda = \max_{1 \leq j \leq 4} \{\Lambda_{h_j}\} = 112/207.$

8.2. Bounded distortion. Choose $\varepsilon_3 = 1$. Since the first three branches are linear they do not influence the choice of distortion constants. As for the last branch, it suffices to show that $\exists \hat{D} > 0$ s.t. $|\ln Jh(x) - \ln Jh(y)| \leq \hat{D}|x - y|, \forall x, y \in (0, 1)$.

By the mean value theorem,
\[ |\ln Jh(x) - \ln Jh(y)| \leq \sup \frac{Jh'}{Jh} |x - y| \leq \sup \frac{T''}{(T')^2} \circ h |x - y| \]
\[ \leq \sup \frac{2h_4(x) + 81/128}{(2(9/16) + 81/128)^2} \leq \frac{25088}{42849}. \]

\textsuperscript{7}The name refers to a class of maps whose graph looks like a “W”. Their main feature is the existence of a periodic turning point which may lead to singular behaviour under perturbation. See [18] for instability and [17] for stability of families of such maps.
Therefore, the bounded distortion condition is satisfied with
\[ \alpha = 1, \tilde{D} = \frac{25088}{42849}, \varepsilon_3 = 1. \]
It follows that \( D = \frac{25088}{19665} \) and \( D/(1 - \Lambda^o) = \frac{25088}{9025} \). So we fix \( a_0 = \frac{25089}{9025} \).

Remark 24. (Restriction to intervals) In our one-dimensional example, in which all \( O_h \) are open intervals, if we can verify hypotheses (3), (4) assuming that \( I, V \) and \( U_\ell, \forall \ell \in U \) are open intervals (a stronger property than being just open sets), then we get a theorem about proper standard pairs whose domains are open intervals. The main reason is that since \( O_h, \forall h \in H_0 \) is an open interval, then so are \( I \cap O_h, T^n(I \cap O_h) \) and \( T^n(I \cap O_h) \cap U_\ell \). Therefore we can repeat all the proofs assuming that standard pairs are supported on open intervals.

In the following we check hypotheses (3), (4) for \( I, V \) being intervals and construct \( U_\ell \) to be intervals \( \forall \ell \). Accordingly our theorem will also be restricted to proper standard pairs whose domains are open intervals.

8.3. Dynamical complexity. Fix \( n_0 = 1 \). If we choose \( \varepsilon_4 = 1/4 \), then any open interval \( I \) of diameter \( \leq \varepsilon_4 \) contains at most one point of discontinuity. By points of discontinuity we mean \( \{9/40, 9/20, 9/16\} \). If \( I \) contains no discontinuities, then the complexity condition holds because the left-hand side of (2.2) is zero. Suppose \( I \) contains a discontinuity. Fix a branch \( h \in H \) s.t. \( m(I \cap O_h) > 0 \). \( T(I \cap O_h) \) is an interval and \( \partial_\varepsilon T(I \cap O_h) \) consists of at most two intervals of length \( \varepsilon \) near the two ends of the interval \( T(I \cap O_h) \). However, if this interval is of the form \((0, p)\) or \((p, 1)\) for some \( p \in (0, 1) \), i.e. if one of its endpoints is 0 or 1, then \( \partial_\varepsilon T(I \cap O_h) \) will consist of at most one subinterval of length \( \varepsilon \) near \( p \). It follows that \( h(\partial_\varepsilon T(I \cap O_h)) \setminus \partial_\Lambda^o I \) is a single subinterval of length \( \leq \Lambda h \varepsilon \) near the discontinuity point that cuts \( I \). A moment of consideration of possible locations of the interval \( I \) in \((0, 1)\) reveals that the only discontinuity point that contributes to the complexity expression, i.e. the expression on the left-hand side of (2.2), is the discontinuity point at \( 9/20 \) and the complexity expression is bounded by
\[
\leq \frac{\Lambda h \varepsilon + \Lambda h_3 \varepsilon}{2 \Lambda \varepsilon} = \frac{1/2 + 1/4}{2 \Lambda} = \frac{621}{896},
\]
where we have used $m(\partial_{\alpha}I) \geq 2\Lambda\varepsilon$. Note that, since $I$ is an interval, $m(\partial_{\alpha}I) = \min\{\text{diam}(I), 2\Lambda\varepsilon\}$; but if $\text{diam}(I) < 2\Lambda\varepsilon$, then $\partial_{\alpha}I = I$ and the numerator of the complexity expression is 0 hence (2.2) is trivially satisfied.

So we choose $\sigma = 621/896$ (which is indeed strictly less than $\Lambda^{-1} - 1 = 95/112$) and then choose $\varepsilon_0$ so that

$$\sigma < e^{-a\sigma_0^3}(\Lambda^{-1} - 1).$$

In fact we choose $\varepsilon_0$ so that $e^{-a\sigma_0^3}(\Lambda^{-1} - 1)$ equals the average of $\Lambda^{-1} - 1$ and $\sigma$: $\varepsilon_0 = (9025/25089)\ln(1520/1381) \approx 0.0345$.

8.4. Divisibility of large sets. Suppose $V$ is an arbitrary open interval with $\text{diam} V \geq \varepsilon_0$. Suppose $V_0 \subset V$ is a set with $\text{diam} V_0 < \varepsilon_0/3$ (so we are taking $\eta = 1/3$). First choose $U_{\varepsilon_0}$ to be an open interval of $\text{diam} U_{\varepsilon_0} \leq \varepsilon_0/3$ containing $V_0$. If on the left or right of $U_{\varepsilon_0}$ there is a piece of $V$ left with $\text{diam} \leq \varepsilon_0/3$, then join it to $U_{\varepsilon_0}$. Now there remains at most two pieces of $V$ of $\text{diam} > \varepsilon_0/3$ to cut into pieces of $\text{diam} \leq \varepsilon_0$. Simply cut the remainder into equal pieces of length $\varepsilon_0/3$ and again if there are pieces left with $\text{diam} \leq \varepsilon_0/3$ join them to the adjacent intervals. With this construction each of the intervals $U_{\varepsilon_0}$ satisfies $\varepsilon_0/3 \leq \text{diam} U_{\varepsilon_0} \leq \varepsilon_0$ and one of them contains $V_0$. Now the result follows from Remark 5 with $C_{\varepsilon_0} = e^{D\varepsilon_0^3}\varepsilon_0^{-1} \approx 181.75$.

8.5. Positively linked. Let us first prove an auxiliary lemma that is useful in estimating the length of the largest component of the image of an interval after it is cut by discontinuities and each piece is expanded. In the following $c$ can be thought of as the length of the interval $I$ which is cut into $n$ pieces of length $\alpha_1, \ldots, \alpha_n$ and then each of the pieces is expanded by, at least, a factor of $c$, respectively.

Lemma 11. Let $N \subset \mathbb{N}$ and $\{z_j\}_{j \in N}$ be such that $z_j > 0, \forall j \in N$, and $\sum_{j \in N} z_j^{-1} < \infty$. Then

$$\min_{\{\alpha_j\}_{j \in N}, \alpha_j \geq 0} \max_{j \in N} \left\{ z_j \alpha_j \right\} \geq \frac{c}{\sum_{j \in N} z_j^{-1}}.$$  

Proof. Simply note that

$$\max\left\{ \frac{\alpha_1}{z_1}, \frac{\alpha_2}{z_2} \right\} \geq \frac{\alpha_1 + \alpha_2}{z_1 + z_2}.$$  

Taking the minimum on both sides over $\alpha_1, \alpha_2 \geq 0$ s.t. $\alpha_1 + \alpha_2 = c$ proves the lemma when $N$ has two elements. The full result follows by induction. \hfill \Box

Let

$$s_H = \max_{j=1,2,3} \{A_{h_j} + A_{h_{j+1}}\} \quad \text{and} \quad \delta_{\text{max}} = \max_{j=1,2,3} \{m(O_{h_j}) + m(O_{h_{j+1}})\}. \quad (8.1)$$

Lemma 12. For every $\delta_1 > 0$ there exist $\tilde{N}_{\delta_1} \in \mathbb{N} \cup \{0\}$ and $\Gamma_{\delta_1} > 0$ s.t. for every interval $J$ with $m(J) \geq \delta_1$, there exist $N \leq \tilde{N}_{\delta_1}$ and a subinterval $J_N \subset J$ such that

(a) $J_N$ is contained in a partition element of $T^N$;

(b) $T^N J_N$ contains a partition element of $T$;

(c) $(T^N J_N)' \leq \Gamma_{\delta_1}^{-N}$.

In fact, $\Gamma_{\delta_1} = 9/40$ and $\tilde{N}_{\delta_1}$ is the least non-negative integer such that $\delta_1/s^\delta_{\tilde{N}_{\delta_1}} \geq \delta_{\text{max}}$.

Proof. Given $\delta_1 > 0$ let $\tilde{N}_{\delta_1}$ be the least non-negative integer such that $\delta_1/s^\delta_{\tilde{N}_{\delta_1}} \geq \delta_{\text{max}}$ and $\Gamma_{\delta_1} = 9/40$. Suppose $J$ is an interval with $m(J) \geq \delta_1$ and it does not
contain a partition element of $T$. By Lemma 11, $J$ contains a subinterval $J_1$, which is contained in a partition element of $T$, and

$$m(TJ_1) \geq \frac{m(J)}{s_H} \geq \frac{\delta_1}{s_H}.$$  

If $m(TJ_1)$ contains a partition element of $T$, then we are done since all three conditions of the lemma are satisfied with $N = 1 \leq \tilde{N}\delta_1$.

If $TJ_1$ does not contain a partition element of $T$, then again by Lemma 11, $TJ_1$ contains an interval $J_2$, which is contained in a partition element of $T$, and

$$m(TJ_2) \geq \frac{m(TJ_1)}{s_H} \geq \frac{\delta_1}{s_H}.$$  

It follows that there exists an interval $J_2 \subset J_1 \subset J$, which is contained in a partition element of $T^2$, and

$$m(T^2J_2) \geq \frac{\delta_1}{s_H}.$$  

This process stops when $T^N J_N$ has length larger than $\delta_{\text{max}}$. By our choice of $\tilde{N}\delta_1$, this happens for some $N \leq \tilde{N}\delta_1$. Since $T^N$ is always $\approx (9/40)^{-N}$, we also have $(T^N J_N)' \leq (9/40)^{-N} = \Gamma_{\delta_1}^{-N}$. □

Now we are prepared to check the positively linked condition (5). Let $\delta := \delta_0$ and $\delta_1 := \delta/3$. Let $C_X = 1$. Divide the unit interval into finitely many subintervals $\{J\}$ of length $\delta/3$ and possibly one last interval of length between $\delta/3$ and $2\delta/3$. Note that $\delta_{\text{max}} = 11/20$, where $\delta_{\text{max}}$ was defined by (8.1). Let $\tilde{N}\delta_1$ be as in Lemma 12. Any interval $J$ belonging to the above finite collection, by Lemma 12, has a further subinterval $J_N$, where $N \leq \tilde{N}\delta_1$, that is contained in a partition element of $T^N$ and, under $T^N$, covers one full partition element $O_\delta$ of $T$. Since for every $h \in \mathcal{H}$, $TO_\delta \supset (0,1/2)$, there is a collection of subintervals $\{J_N \subset J\}$ each of whose elements is contained in a partition element of $T^{N+1}$ and, under $T^{N+1}$, cover $(0,1/2)$. Note that $N + 1 \leq \tilde{N}\delta_1 + 1$, so for condition (5) we can take

$$N_\delta = \tilde{N}\delta_1 + 1 = \max \left\{ \frac{\ln \frac{\delta_1}{s_{\text{max}}}}{\ln s_H}, 0 \right\} + 1 = 57.$$  

The subintervals $\{J_N\}$ constitute the collection $Q_N$ and we take $\Omega = (0,\epsilon_0/3) \subset (0,1/2)$. By Remark 11, $\Omega$ is a $C_X$-good overlap set with $C_X = 1$.

- The $\delta$-density condition is satisfied because every $\delta$-regular set $I$ contains an open interval of length $\delta$ which in turn contains at least one interval $J$ of length between $\delta/3$ and $2\delta/3$. The interval $J$ in turn contains an element $J_N$ of $Q_N$, by construction.
- For every $Q, \tilde{Q} \in Q_N$, $T^N Q \cap T^N \tilde{Q}$ contains the interval $\Omega = (0,\epsilon_0/3)$, by construction and clearly $m(\Omega) = \epsilon_0/3$. Recall that since $n_0 = 1$, $M = 1$.
- For every $Q \in Q_N$, $N \leq N_\delta$ and $h \in \mathcal{H}$ with $Q \subset O_h$ we have

$$\inf_{T^N Q} Jh \geq \inf_{x \in O_h} 1/(T^N)'(x) \geq (9/40)^N \geq (9/40)^{N_\delta}.$$  

Using the quantities above we get $1 - \gamma_2 \approx 10^{-41}$, which leads to a $1/2$-mixing time of $t_\ell \approx 10^{41}$ for $(a_0,\epsilon_0, B_0)$-proper standard pairs. This mixing time depends most significantly on the value of the lower bound on $\Gamma_N \geq (9/40)^{N_\delta}$. If by numerical simulation, or by considering higher iterates of the map we find out that $N_\delta = 20$ suffices, then using this value gives a $1/2$-mixing time of $\approx 10^{17}$.

Remark 25 (Comparison of constants). Let us now briefly comment on the result that one obtains by using Hilbert metric contraction as done in [21, Theorem 4]. In [21, Theorem 4], the significant factor in the bound on correlations is $\approx (1 - 10^{-8})^n$. 
This leads to a \(1/2\)-mixing time of roughly \(10^8\), which is significantly better than \(10^{40}\) or even \(10^{17}\). The main reason for this is the additional information on the global regularity of functions under iterations of the transfer operator \(L\). Indeed, [21] uses the facts that the transfer operator of this dynamical system preserves the space of functions of bounded variation (BV) and that the Lasota-Yorke inequality holds in this space. Using BV and Lasota-Yorke inequality one can show that the iterations of densities of bounded variation under \(L\) have uniformly bounded variation. This in turn implies that they have a uniform lower bound on some interval hence by the topological exactness (and uniform bounds on \(T\)) they have a uniform lower bound on the whole space. Then the contraction in the Hilbert metric is used to find the rate of decay of correlations. However, one could just as well use the coupling approach of this paper to couple densities that overlap on the whole space and have a uniform lower bound. So it is not the Hilbert metric contraction that improves the estimate, but the information on the global regularity of iterates of densities under the transfer operator. Note that the specific notion of regularity is important: a BV function in two dimensions need not contain an open set in its support (for a simple example see [20]). In this paper we are using coupling without information on global regularity, which makes our approach more flexible. However, as we pointed out, one can use coupling + global regularity information and get the same results as using Hilbert metric contraction + global regularity information. At least in the piecewise expanding setting Hilbert-metric contraction does not seem to have an advantage over coupling. In a much narrower setting this observation (equivalence of coupling and Hilbert metric contraction) was already made in [38].

9. Example 2: A non-Markov map of \(\mathbb{R}\)

In this section we verify our assumptions for a piecewise expanding map of \(X = \mathbb{R}^+ = (0, \infty)\), where \(d\) is the usual metric and the underlying measure \(m\) is the Lebesgue measure. Take \(\varepsilon_1 = \infty\). Fix \(t = 0.1\). The map \(T : (0, \infty) \to \mathbb{R}^+\) is defined on a countable partition \(\mathcal{P} = \{O_1, O_2, \ldots\}\), where

\[
O_{2k-1} = (k-1, k-t) \text{ and } O_{2k} = (k-t, k), \quad \forall k \in \mathbb{N}.
\]

\(T\) is defined by

\[
T(x) = \begin{cases} 
(10 + 2^{-k})(x - k + 1), & x \in O_{2k-1}, k \in \mathbb{N}, \\
\frac{1}{x-t}, & x \in O_{2k}, k \in \mathbb{N}.
\end{cases}
\]

Note that \(T\) is piecewise increasing and has infinitely many different images: \(\forall k \in \mathbb{N}\) the images are

\[
TO_{2k-1} = (T(k-1), T(k-t)) = (0, (10 + 2^{-k})(1-t)) \\
TO_{2k} = (T(k-t), T(k)) = (1/t, \infty).
\]

Also note that \(T\) is not surjective. In particular no point maps into the interval \((9.45, 10)\).

9.1. Uniform expansion. Take \(\varepsilon_2 = \infty\). In terms of inverse branches we have

\[
h_{2k-1}(x) = \frac{x}{10 + 2^{-k}} + k - 1 \quad \text{and} \quad h_{2k}(x) = -\frac{1}{x} + k.
\]

So

\[
h_{2k-1}'(x) = \frac{1}{10 + 2^{-k}} < \frac{1}{10} \quad \text{and} \quad h_{2k}'(x) = \frac{1}{x^2} < t^2.
\]

It follows that \(\Lambda = 1/10 \geq \Lambda_{h_{2k-1}}\) and \(\Lambda_{h_{2k}} \leq t^2\).
9.2. Bounded distortion. Take $\varepsilon_3 = \infty$. The odd branches are linear, we check distortion for the even branches. Suppose $x, y \in TO_{2k} = (1/t, \infty)$, then

$$|\ln h'_{2k}(x) - \ln h'_{2k}(y)| = 2|\ln x - \ln y| \leq 2t|x - y|,$$

where in the last inequality we have used the mean value inequality and $x, y > 1/t$.

We conclude that the bounded distortion condition is satisfied with $\tilde{D} = 2t$ and $\alpha = 1$. So $D = 20t/9$ and we can take $a_0 > 200t/81$. For our choice of $t = 0.1$, we take $a_0 = 21/81 > 20/81$.

9.3. Dynamical complexity. Choose $n_0 = 1$ and $\varepsilon_4 = 1/2$. Any open interval $I$ of diam $I \leq \varepsilon_4$ will intersect at most three adjacent partition elements of the form $O_{2k-1}, O_{2k}, O_{2k+1}$ for some $k$. This is the worst case for the dynamical complexity condition. In this case $I$ is cut into three pieces at the two cut-points: $k - t$ and $k$. Considering the $\varepsilon$-boundary of the image of the pieces and pulling them back to $I$, we get the following bound on the numerator of the complexity expression (left-hand side of (2.2))

$$\leq \Lambda h_{2k-1}\varepsilon + \Lambda h_{2k}\varepsilon + \Lambda h_{2k+1}\varepsilon.$$

Note that the left part of the cut point at $k$ does not contribute to the complexity expression because its image has empty $\varepsilon$-boundary. So the complexity expression is bounded by

$$\frac{\Lambda h_{2k-1}\varepsilon + \Lambda h_{2k}\varepsilon + \Lambda h_{2k+1}\varepsilon}{2\Lambda\varepsilon} \leq \frac{1/10 + t^2 + 1/10}{2/10} = 1 + 5t^2 = \frac{21}{20}.$$

So we choose $\sigma = 21/20$, which is indeed strictly less than $\Lambda^{-1} - 1 = 9$. We also have $\sigma < 9e^{-a_0\varepsilon_0} = 9e^{-1/10} \approx 8.14$. So we can take $\varepsilon_0 = \varepsilon_4 = 1/2$.

9.4. Divisibility of large sets. Just as in the previous example, the result follows from Remark 5 with $C_{x_0} = e^{D \varepsilon_0^2} 6\varepsilon_0^{-1} = 12e^{1/10} \approx 13.26$.

9.5. Positively linked. Now we need to find a collection of $\delta = \delta_0$-dense sets that interact in a finite time $N \leq N_0$ that is uniform for every pair of such intervals and the overlaps have uniform lower bound $\Delta$ on their measure. $\delta_0$ can be calculated using the formulas in Section 2 and one gets $\delta_0 \approx 0.0134$. 

Figure 2. The graph of the map $T$ from Example 2.
Let
\[ s_H := \frac{1}{10} + t^2 = 0.11. \]

**Lemma 13.** For every \( \delta_1 > 0 \) there exist \( N_{\delta_1} \in \mathbb{N} \cup \{0\} \) and \( \Gamma_{\delta_1} > 0 \) s.t. for every interval \( J \) with \( m(J) \geq \delta_1 \), there exists \( N \leq N_{\delta_1} \) and a subinterval \( J_N \subset J \) such that

(a) \( J_N \) is contained in a partition element of \( T^N \).

(b) \( T^N J_N \) contains a partition element of \( T \).

(c) \( (T^N|_{J_N})' \leq \Gamma_{\delta_1}^N \).

In fact, we can choose any \( s \in (0, \min\{1 - s_H, 1/(\delta_1 \sqrt{10 + 2^{-1}})\}) \) and then take \( \Gamma_{\delta_1} = (s\delta_1)^{-2} \) and \( N_{\delta_1} \) the least non-negative integer such that
\[ \frac{\delta_1}{s_{N_{\delta_1}}^2} \left( 1 - s \sum_{j=0}^{N_{\delta_1} - 1} s_H \right) \geq 1. \]

**Proof.** Suppose \( J \) is an interval with \( m(J) \geq \delta_1 \). If \( J \) contains a partition element of \( T \) then \( N \) can be taken to be 0 and \( J_0 \) to be equal to the partition element contained in \( J \). Also if \( m(J) \geq 1 \) then \( J \) necessarily contains a partition element of \( T \) and the same argument applies. So let us assume that \( J \) is an arbitrary interval with \( \delta_1 \leq m(J) < 1 \) and it does not contain a partition element of \( T \). Set \( s = 0.6 \).

Let \( K_s(k) = (k - s\delta_1, k) \), \( \forall k \in \mathbb{N} \), and let \( K_s = \bigcup_{k \in \mathbb{N}} K_s(k) \). \( K_s \) is the union of one-sided intervals of length \( s\delta_1 \) where \( T' \) is unbounded at the right endpoint of each interval. \( J \setminus K_s \) is a union of at most two intervals and \( m(J \setminus K_s) \geq m(J) - s\delta_1 \).

If \( J \) does not intersect \( K_s \), then \( J \) can only contain one of the discontinuities at \( k - t \), where the derivative of \( T \) on its left side is \( \geq 10 \) and on its right side \( \geq 1/t^2 \). Therefore, by Lemma 11, \( J \) contains an interval \( J_1 \), which is contained in an element of \( \mathcal{P} \), and \( m(T J_1) \geq \frac{m(J)}{1/10 + t^2} = \frac{m(J)}{s_H} \).

If \( J \) intersects \( K_s(k) \), then \( J \setminus K_s(k) \) is a union of at most two intervals which lie on the left and right of \( K_s(k) \), namely \( J \cap (O_{h2k} \setminus K_s(k)) \) and \( J \cap O_{h2k+1} \). The derivatives of \( T \) on these two sets are \( \geq 10 \) and \( \geq 1/t^2 \). It follows, again by Lemma 11, that \( J \setminus K_s(k) \) contains an interval \( J_1 \), which is contained in a partition element of \( T \), and \( m(T J_1) \geq \frac{m(J \setminus K_s(k))}{s_H} \geq \frac{m(J) - s\delta_1}{s_H} \).

The maximum derivative \( T' \) on \( J \setminus K_s(k) \), and hence on \( J_1 \), is \( \max\{10 + 2^{-1}, 1/(s\delta_1)^2\} = 1/(s\delta_1)^2 \).

Note that if \( m(T J_1) \geq 1 \), then \( T J_1 \) necessarily contains \( O_h \) for some \( h \in \mathcal{H} \). Furthermore \( (T|_{J_1})' \leq \frac{1}{(s\delta_1)^2} \). So we have proved our claim with \( N = 1 \).

If \( m(T J_1) < 1 \) and \( T J_1 \) does not contain a partition element of \( T \), we repeat the above argument with \( J \) replaced by \( T J_1 \). In conclusion there exists an interval \( J_2 \subset T J_1 \), which is contained in a partition element of \( T \), and \( m(T J_2) \geq \frac{m(T J_1) - s\delta_1}{s_H} \geq \frac{m(J) - s\delta_1}{s_H} \).

\( (T|_{J_2})' \leq \frac{1}{(s\delta_1)^2} \).
It follows that there exists an interval $J_2 \subset J_1 \subset J$, which is contained in a partition element of $T^2$, such that
\[
\mathfrak{m}(T^2 J_2) \geq \frac{\mathfrak{m}(J) - s_0(1 + s_H)}{s_H}
\]
\[
(T^2 | J_2)' \leq \sup(T | J_2) \sup(T | J_2)' \leq \sup(T' | J_2) \sup(T | J_2)' \leq \left(\frac{1}{(s_0/2)^2}\right)^2.
\]
By induction one can show that there exists an interval $J_N \subset J$, which is contained in a partition element of $T^N$, and
\[
\mathfrak{m}(T^N J_N) \geq \frac{\mathfrak{m}(J) - s_0 \sum_{j=0}^{N-1} s_H^j}{s_H^N} \geq \frac{\delta_1 \left(1 - s \sum_{j=0}^{N-1} s_H^j\right)}{s_H^N},
\]
\[
(T^N | J_N)' \leq \left(\frac{1}{(s_0/2)^2}\right)^N.
\]
Clearly for some finite $N$, that does not depend on $J$ except through $\delta_1$, we will have $\mathfrak{m}(T^N J_N) \geq 1$. We assumed $s < 1 - s_H$ so that $1 - s \sum_{j=0}^{N-1} s_H^j > 0$. \hfill $\square$

Now divide $X = (0, \infty)$ into equal intervals $\{J\}$ of length $\delta_1 = \delta_0/3$. Note that by Lemma 13, each interval $J$ has a further subinterval $J_N$, which is contained in a partition element of $T^N$, and under $T^N$ covers one full partition element $O_h \in \mathcal{P}$. At this point one can calculate $N$. Recall that $\delta_0$ is calculated by the formulas of Section 2. My calculation gives $N = 3$. Since for every $h \in \mathcal{H}$, $T^2 O_h \supset (0, \infty) \setminus (9, 10)$, there is a collection of subintervals $\{J_N \subset J\}$, each of which is a subset of an element of $\mathcal{P}^{N+2}$ and covers $(0, \infty) \setminus (9, 10)$ under $T^{N+2}$. Furthermore, it is easy to choose $J_N$ so that $T^{N+1} J_N$ does not intersect $K_0$. This is needed to ensure that $(T^{N+1})'$ is bounded on $J_N$.

Let $N_0 = N + 2 = 5$. The sub-subintervals $\{J_N\}$ constitute the collection $\mathcal{Q}_{N_0}$ and we take the overlap set $\Omega = (0, \varepsilon_0/3) = (0, 1/6)$. By Remark 11, $\Omega$ is a $C_X$-good overlap set with $C_X = 1$.

- The $\delta$-density condition is satisfied because every $\delta$-regular set $I$ contains an open interval of length $2\delta$ which in turn contains at least one interval $J$ of length $\delta/3$. The interval $J$ in turn contains an element $J_N$ of $\mathcal{Q}_{N_0}$, by construction.
- For every $Q, \tilde{Q} \in \mathcal{Q}_{N_0}$, $T^{N_0} Q \cap T^{N_0} \tilde{Q}$ contains the interval $\Omega = (0, \varepsilon_0/3)$ by construction, and $\mathfrak{m}(\Omega) = \varepsilon_0/3 = 1/6$.
- For every $Q \in \mathcal{Q}_{N_0}$, $h \in \mathcal{H}^{N_0}$ with $Q \subset O_h$ we have
  \[
  \inf_{T^{N_0} Q} J_h \geq \inf_{x \in O_h} 1/(T^{N_0})'(x) \geq (s_0/2)^{N_0}.
  \]
  Using the quantities above, with $s = 0.6$, we get
  \[
  1 - \gamma_2 \approx 10^{-29},
  \]
  which leads to a $1/2$-mixing time of $t_* \approx 10^{29}$ for $(a_0, \varepsilon_0, B_0)$-proper standard pairs.

10. Example 3: A 2D example and construction of a Tower

In this section we describe a two-dimensional piecewise expanding map with a countably infinite partition that is neither Markov nor conformal. After the description of the map we check conditions (1)-(4). Then we proceed to induce the 2D piecewise expanding map, with exponential tails, to a Gibbs-Markov map with
finitely many images. Statistical properties for our example can then be deduced in a standard manner [37]. If one is interested in an explicit bound on the mixing time of the system, one would need to also check the condition (5). We will not pursue this route in the current example.

Suppose $s \in (0, 1)$ is sufficiently small (to be determined later) and let $W := W(s) = (1/5) \sum_{i=1}^{\infty} i^{-(s+1)}$ be a normalization factor. For every integer $i \geq 2$ let $A_i = (0, (5W^{i+1} - 1) \times (0, 5^{-i})$ and define $\tilde{T}_i : A_i \to \mathbb{R}^2$ by

$$T_i(x, y) = (5W^{i+1}x(1 + y), 5^i y)$$

Let $X = \bigcup_{i \geq 2} \tilde{T}_i A_i$, $d$ the metric on $\mathbb{R}^2$ induced by the 2-norm, and $m$ the Lebesgue measure. Take $\varepsilon_1 = 2$. For the current example, if $A \subset X \subset \mathbb{R}^2$,

$$\partial A := \text{cl}_{\mathbb{R}^2} A \cap \text{cl}_{\mathbb{R}^2}(\mathbb{R}^2 \setminus A).$$

This is an important difference with respect to our previous examples.

For $i \geq 2$ and $1 \leq j \leq 5^i$ let $v_{i,j}$ be the vector

$$v_{i,j} = \left(1 - \frac{1}{5W} \sum_{k=1}^{i} k^{-(s+1)}, j - 1 \right)$$

and define

$$O_{i,j} = A_i + v_{i,j}$$

For $i = 1$, and $1 \leq j \leq 5$ let

$$O_{1,j} = \left\{(x, y) \in X : \left(\frac{j - 1}{5} < y < \frac{j}{5}\right) \right\}$$

Note that the collection of open sets $\{O_{i,j}\}$ defined as above forms a (mod 0)-partition of $X$ into open sets. Now we define the map $T : X \cap \mathbb{R}^2$ by defining it on each $O_{i,j}$.

If $i \geq 2$, $1 \leq j \leq 5^i$ and $(x, y) \in O_{i,j}$, then

$$T(x, y) = T_i((x, y) - v_{i,j}).$$

If $i = 1$, $1 \leq j \leq 5$ and $(x, y) \in O_{1,j}$ then

$$T(x, y) = \left(\frac{1}{5W} + \frac{1}{5}\right)^{-1}(x - 1 + \frac{1}{5W}), 5(y - j + 1)\right).$$

See Figure 3 for a depiction of the partition elements and the action of the map $T$.

Note that, for $i = 1$ and $1 \leq j \leq 5$ we have

$$DT(x, y) = \begin{bmatrix} \frac{1}{5W} & 0 \\ 0 & 5 \end{bmatrix} \text{ and } DT^{-1}(x, y) = \begin{bmatrix} \frac{1}{5W} & 0 \\ 0 & \frac{1}{5} \end{bmatrix},$$

while for $i \geq 2$ and $1 \leq j \leq 5^i$, we have

$$DT(x, y) = \begin{bmatrix} 5W^{i+1}(1 + y) & 5W^{i+1}x \\ 0 & 5^i \end{bmatrix} \text{ and } DT^{-1}(x, y) = \begin{bmatrix} \frac{1}{5W^{i+1}(1+y)} & 0 \\ 0 & \frac{1}{5^i} \end{bmatrix}.$$
10.1. **Uniform expansion.** Consider $T^iO_{i,j}$ for some $i \geq 1$ and $1 \leq j \leq 5^i$. For every $z_1, z_2 \in T^iO_{i,j}$ the line segment joining $z_1$ and $z_2$ is contained in $T^iO_{i,j}$. Therefore, by the mean value inequality we have, $\forall z_1, z_2 \in T^iO_{i,j},$

\[
\|h(z_2) - h(z_1)\|_2 \leq \sqrt{d} \|h(z_2) - h(z_1)\|_0 \leq \sqrt{d} \sup_{z \in T^iO_{i,j}} \|Dh(z)\|_0 \|z_2 - z_1\|_0
\]

\[
\leq \sqrt{2} \sup_{h(z) \in O_{i,j}} \|DT^{-1}(h(z))\|_0 \|z_2 - z_1\|_2
\]

\[
\leq \Lambda_{i,j} \|z_2 - z_1\|_2,
\]

where $\|\cdot\|_0$ denotes the sup-norm and

\[
\Lambda_{1,j} = \sqrt{2} \max \left\{ \frac{1}{5^j}, \frac{1}{5^j} \frac{1 + W}{5W}, \right\}, 1 \leq j \leq 5
\]

\[
\Lambda_{i,j} = \sqrt{2} \max \left\{ \frac{1}{5^j}, \frac{1}{5^j} \frac{1 + W^i}{5W^i}, \right\}, i \geq 2, 1 \leq j \leq 5^i.
\]

Now we fix $s$ small enough so that $W = W(s) > 10$. Then, $\Lambda_{1,j} \leq (1.1\sqrt{2})/5$ and $\Lambda_{i,j} \leq (s+1)/20 \leq 1/10, \forall i \geq 2, 1 \leq j \leq 5^i$. Therefore we can take $\Lambda = (1.1\sqrt{2})/5$. Note that we can take $\varepsilon_2 = \infty$.

10.2. **Bounded distortion.** Suppose $z_1, z_2 \in T^iO_{i,j}$ for some $i \geq 1$ and $1 \leq j \leq 5^i$. Suppose $h(z_1) = (x_1, y_1)$ and $h(z_2) = (x_2, y_2)$. We have
\[ Jh(z_1) = \left| \det DT^{-1}(h(z_1)) \right| = \frac{1}{5^{s+1}W^{s+1}(1 + y_1)}, \]

\[ Jh(z_2) = \frac{1}{5^{s+1}W^{s+1}(1 + y_2)}. \]

Therefore

\[ \left| \ln \frac{Jh(z_1)}{Jh(z_2)} \right| = \left| \ln(1 + y_2) - \ln(1 + y_1) \right| \leq C \| z_1 - z_2 \|_2, \]

where \( C = \sup_y 1/(1 + y) \leq 1. \) So comparing to (2.1), we can take, \( \varepsilon_3 = \infty, \tilde{D} = 1 \) and \( \alpha = 1. \)

### 10.3. Complexity

We need to check that there exists \( \varepsilon_4 > 0 \) such that for every open set \( I, \) \( \text{diam } I \leq \varepsilon_4 \) and \( \varepsilon < \varepsilon_4, \)

\[
\sum_{\{h \in \mathcal{H} : m(I \cap O_h) > 0\}} \frac{m(h(\partial_T(I \cap O_h)) \setminus \partial_{\Lambda^2} I)}{m(\partial_{\Lambda^2} I)} \leq \sigma < (\Lambda^{-1} - 1). \tag{10.1}
\]

Recall that in this section by \( \partial A \) we mean the boundary in \( \mathbb{R}^2 \) (not in \( X \)).

In order to estimate each term of the complexity expression we need to map the set \( I \cap O_h \) forward, consider its \( \varepsilon \)-boundary and map it back using the inverse branch \( h. \) Since in our example, each branch can be extended to the boundary in the sense of Remark 4, it is clear that after pulling back, we get a set which is contained in the \( \Lambda \varepsilon \)-boundary of \( I \cap O_h. \) However, this is not enough for our purposes, because there are exponentially many horizontal boundaries and \( \Lambda_h = \Lambda_{i,j} \) only decreases polynomially (in \( i \)). So we need to estimate the contributions from horizontal boundaries more carefully. This can be done using the fact that in our example horizontal boundaries of \( O_h \) map to horizontal boundaries of \( X \) and \( T \) is a skew-product that contracts vertical distances by an exponential factor. Hence the contribution from each horizontal boundary is exponentially small. Note that the skew-product nature of \( T \) is crucially used here.

Let us now formally describe how to estimate \( m(h(\partial_T(I \cap O_h)) \setminus \partial_{\Lambda^2} I). \) First we describe how to split the set \( h(\partial_T(I \cap O_h)) \) so that we can take advantage of the exponential contraction in the vertical direction. Recall that by Remark 4,

\[
\partial_T(I \cap O_h) \subset T\partial(I \cap O_h).
\]

Also, it is a fact that

\[
\partial(I \cap O_h) \subset (\partial I \cap \text{cl } O_h) \cup (\text{cl } I \cap \partial O_h).
\]

It follows that

\[
\partial_T(I \cap O_h) = \{ y \in T(I \cap O_h) : d(y, \partial_T(I \cap O_h) < \varepsilon \}
\]

\[
\subset \{ y \in T(I \cap O_h) : d(y, T(\partial I \cap \text{cl } O_h) \cup (\text{cl } I \cap \partial O_h)) < \varepsilon \}
\]

\[
\subset \{ y \in T(I \cap O_h) : d(y, T(\partial I \cap \text{cl } O_h)) < \varepsilon \} \cup
\]

\[
\{ y \in T(I \cap O_h) : d(y, T(\text{cl } I \cap \partial O_h)) < \varepsilon \}. \tag{10.2}
\]

The second term of the right-hand side can further be split into two horizontal and two vertical parts. We would like to obtain a better estimate on the horizontal parts and the following lemma serves this purpose.

**Lemma 14.** Suppose \( h \in \mathcal{H}^n \) is an inverse branch and \( H^1_h \) is the bottom horizontal boundary of \( O_h. \) Then

\[
\{ x \in I \cap O_h : d(Tx, T(\text{cl } I \cap H^1_h)) < \varepsilon \} \subset \{ x \in I \cap O_h : d(x, \text{cl } I \cap H^1_h) < 5^{-n}\varepsilon \}. \tag{10.3}
\]
Proof. This follows from the fact that $T(cl I \cap H_h)$ is contained in a horizontal line (bottom edge of $X$) and that $h \in \mathcal{H}^n$ contract vertical distances by a factor of $5^{-n}$. □

Suppose $H^2_h, V^1_h, V^2_h$ are the top, left and right boundaries of the rectangle $O_h$. It follows from (10.2), (10.3) and condition (1)

\[
\begin{align*}
h(\partial T(I \cap O_h)) &\subset \{ x \in I \cap O_h : d(Tx, T(\partial I \cap cl O_h)) < \varepsilon \} \\
&\quad \cup \{ x \in I \cap O_h : d(Tx, T(cl I \cap H^{1,2}_h)) < \varepsilon \} \\
&\quad \cup \{ x \in I \cap O_h : d(x, \partial I \cap cl O_h) < \Lambda_h \varepsilon \} \\
&\quad \cup \{ x \in I \cap O_h : d(x, cl I \cap V^{1,2}_h) < 5^{-n} \varepsilon \} \\
&\quad \cup \{ x \in I \cap O_h : d(x, cl I \cap V^{1,2}_h) < \Lambda_h \varepsilon \}.
\end{align*}
\]  
(10.4)

Now that we have isolated the exponential contribution of horizontal boundaries, we are almost ready to estimate $m(h(\partial T(I \cap O_h)) \setminus \partial \Lambda \varepsilon I)$. We just need one last ingredient – a lemma from [5].

Lemma 15 (Sublemma C.1 of [5]). Suppose $I$ is a non-empty measurable bounded subset of the plane and $E$ is a straight line cutting $I$ into left and right parts $I_l$ and $I_r$. Then $\forall \varepsilon \geq 0$ and $0 \leq \xi \leq 1$, we have

\[
\begin{align*}
m(\{ x \in I_l : d(x, E) \leq \xi \varepsilon \} \setminus \{ x \in I : d(x, \partial I) \leq \varepsilon \}) &\leq \\
\xi m(\{ x \in I_r : d(x, \partial I) \leq \varepsilon \}).
\end{align*}
\]  
(10.5)

Proof of eq. (10.1). Consider a set $I$ of diam $I \leq \varepsilon_4$, where $\varepsilon_4$ is sufficiently small and will be determined shortly. By a corner point we mean a point at which more than two (i.e. three or four) partition elements meet.

We consider two cases:

(1) $\text{cl } I$ contains at most one corner point.

![Figure 4: Case 1: the open set $I$ and boundaries of partition elements (solid lines). The dashed line is the continuation of the horizontal line segment that ends inside $I$.](image)

In this case at the corner point we have one vertical line and one horizontal line intersecting (Figure 4). We continue each line smoothly until it intersects the boundary of $I$.

This way we get two lines that intersect $I$ completely and it follows from (10.5) that each one contributes at most as much as the corresponding boundary of $I$. So, recalling (10.4), the overall contribution of such boundaries (one straight vertical line and one straight horizontal line) is

\[
\leq \frac{2m(\partial \Lambda \varepsilon I)}{m(\partial \Lambda \varepsilon I)} = 2.
\]

So the condition (10.1) is satisfied.
(2) \( cl I \) contains two or more corner points.

In this case the smaller the diameter of \( I \), the closer it must be to the line of accumulation of \( \{ O_{i,j} \} \). So \( I \) may intersect only \( O_{i,j} \) for sufficiently large \( i \geq i_0 \). In this case the set \( I \) can intersect infinitely many singularity lines (lines consisting of boundaries of partition elements). Our singularity lines are of two types: the vertical lines which cross through \( I \); the horizontal lines infinitely many of which may terminate inside \( I \). Please recall (10.4) showing how the vertical and horizontal singularity lines contribute to the complexity expression.

(2.1) **Vertical strips.**

![Figure 5](image)

**Figure 5.** Case 2.1: the open set \( I \) and infinitely many vertical singularity lines crossing it. Each line consists of pieces of boundaries of partition elements. The horizontal singularity lines are not shown in the figure to prevent clutter.

In this case, using Lemma 15 with \( \xi = \Lambda_n / \Lambda \) and \( \varepsilon \) replaced by \( \Lambda \varepsilon \), the complexity expression is bounded by

\[
\leq \frac{m(\partial \Lambda \varepsilon I)}{m(\partial_{\Lambda \varepsilon} I)} \sum_{n=i_0}^{\infty} \frac{\Lambda_n / \Lambda}{\Lambda^{-1} \sum_{n=i_0}^{\infty} \Lambda_n}, \tag{10.6}
\]

where \( \Lambda_n := \sup_{1 \leq j \leq 5^n} \Lambda_{n,j} \).

Since \( \sum_{n=1}^{\infty} \Lambda_n < \infty \), the above expression can be made arbitrarily small by making \( i_0 \) sufficiently large. In turn, \( i_0 \) can be made arbitrarily large by choosing \( \varepsilon_4 \) sufficiently small. So by choosing \( \varepsilon_4 \) sufficiently small we can make (10.6) smaller than the contribution of case 1 where \( I \) contains at most one corner point.

(2.2) **Horizontal strips.** Let \( B_I \) be a ball such that \( I \subset B_I \) and \( \text{diam} B_I = \text{diam} I \). The horizontal strips (horizontal singularity lines and their \( \Lambda_{h,\varepsilon} \)-boundaries) that intersect \( B_I \) are strips of variable width around straight horizontal lines \( \{ H_{k,j} \} \) that go all the way across \( B_I \) or terminate inside \( B_I \) on a vertical singularity line \( V_k \). Let \( \{ H_{i,j} \}_{j=N_1}^{N_2} \) be the collection of horizontal segments that terminate on the rightmost vertical line \( V_{i_0} \) that intersects \( B_I \). Consider a uniform strip of width \( 5^{-i_0} \varepsilon \) around each \( H_{i_0,j}, j = N_1(i_0), \ldots, N_2(i_0) \). The measure of these large strips of uniform width bounds the measure of all the Horizontal strips that intersect \( B_I \). This is because the original strips have variable width (their widths decrease exponentially as they approach the accumulation line of partition elements. See Figure 6 which depicts how the smaller strips of variable width can be combined to form a large strip of fixed width. In the figure it is assumed that the map is doubling in the vertical direction just for the clarity of the figure. We must point out that to make this argument rigorous one needs to imitate the proof of Lemma 15. This is not difficult so we omit the technical details.

The measure of each such “effective strip” is at most as much as \( 5^{-i_0} / \Lambda \) times the \( \Lambda \varepsilon \)-boundary of \( I \) by Lemma 15. So the measure of all the horizontal strips is
bounded by
\[ \leq (N_2(i_0) - N_1(i_0))(5^{-i_0}/\Lambda)\mathcal{m}(\partial_\Lambda I), \]
where \((N_2(i_0) - N_1(i_0)) \leq C \text{diam}(B_I)5^{i_0}\) since there are 5\(^{i_0}\) equally spaced horizontal singularity lines that cross \(X\) and terminate on \(V_{i_0}\). So in this case the complexity expression is bounded by
\[ \leq \frac{C \text{diam}(B_I)\Lambda^{-1}\mathcal{m}(\partial_\Lambda I)}{\mathcal{m}(\partial_\Lambda I)} \leq CA^{-1} \text{diam}(B_I) \]
Clearly this quantity can also be made arbitrarily small by choosing \(\varepsilon_4\) sufficiently small.

\[ \Box \]

Figure 6. Case 2.2: Merging horizontal boundaries of segments that terminate inside \(I\).

Now we check hypothesis (4) and conclude that the growth lemma holds. Then we proceed to induce \(T\), with exponential tails, to a Gibbs-Markov map with finitely many images.

10.4. Divisibility of large sets. This condition follows from Remark 5. Recall that distortion bound holds for \(\varepsilon_4 = \infty\).

10.5. Inducing partition. In this subsection we check conditions (7) and (8) simultaneously.

Proof of conditions (7) and (8). Let \(c = 0.01\) and let \(S = \{S_j\}_j\) denote a grid of open squares in \(\mathbb{R}^2\) whose elements have sides parallel to the horizontal and vertical axes and have side-length \(c\delta_0\). Let \(R = \{S \cap X : S \in S\}\). Since \(X\) is a bounded subset of \(\mathbb{R}^2\), \(R\) has only finitely many elements.

Item (1) of (7) holds by direct calculation. Indeed, \(\mathcal{m}(\partial R) \leq 4\varepsilon c\delta_0\) for every \(R \in R\), except for elements \(R = S \cap X\) where \(S\) intersects the right boundary of \(X\). This boundary is Lipschitz so \(\mathcal{m}(\partial R) \leq (4 + L_0)\varepsilon c\delta_0\), where \(L_0\) is the Lipschitz constant.

Before we prove item (2) and item (3) of (7), let us remark that since the partition \(R\) is finite, it suffices to prove these statements with a choice of constants
\(c_R, C_R > 0\) that depend on the partition element \(R \in \mathcal{R}\) because we can then choose \(c_R = \min\{c_R : R \in \mathcal{R}\}\) and \(C_R = \max\{C_R : R \in \mathcal{R}\}\).

To prove item (2) of (7), suppose \(I \subset X\) is a \(\delta_0\)-regular set. Then it contains a ball \(B_{\delta_0}(x, \delta_0)\) of radius \(\delta_0\). Let \(S \subset \mathcal{S}\) be a square containing the center of the ball \(B_{\delta_0}(x, \delta_0)\), possibly on its boundary. Since \(\text{diam} S < c\delta_0\), \(S \subset B_{\delta_0}(x, c\delta_0)\). Therefore \(R := S \cap X \subset B_{\delta_0}(x, c\delta_0) \subset I\). To see (7.5), note that \(\exists c_R \in (0, 1)\) such that

\[
\mathbf{m}(B_{\delta_0}(x, c\delta_0)) < (1 - c_R)\mathbf{m}(B_{\delta_0}(x, \delta_0)).
\]

Therefore,

\[
\mathbf{m}(R) \leq \mathbf{m}(B_{\delta_0}(x, c\delta_0)) < (1 - c_R)\mathbf{m}(B_{\delta_0}(x, \delta_0)) \leq (1 - c_R)\mathbf{m}(I).
\]

It follows that \(\mathbf{m}(I \setminus R) \geq c_R\mathbf{m}(I)\). (7.6) holds as a consequence of Lemma 15 since \(R\) has at most four sides that lie in \(X\) and each one can be continued as a straight line to cross \(I\) and the \(\varepsilon\)-boundary of each contributes as much as the \(\varepsilon\)-boundary of \(I\), so \(\mathbf{m}(\partial_e(I \setminus R) \setminus \partial_e I) \leq 4\mathbf{m}(\partial_e I)\).

Now let us show item (3) of (7). Let \(D\) denote the left vertical side of \(X\) on which partition elements \(\{O_{i,j}\}\) of \(T\) accumulate. Let \(Z \subset \mathcal{R}\) be an open rectangle that contains (perhaps on its boundary) the midpoint \(l_0\) of the segment \(L\). Since \(\eta = 1/6\) and \(\delta_0 \leq \delta_0\), we have \(\text{diam} Z \leq c\delta_0 \leq \eta e_0\), so (7.7) is satisfied. Let \(Z' = B_{\delta_0}(l_0, 2c\delta_0)\), then (7.13) and (7.14) of (8) are also satisfied.

To check (7.9), suppose \(I \subset X\) is a \(\delta_0\)-regular set and \(B_{\delta_0}(x, \delta_0) \subset I\). There exists a translation \(Z_{\delta_0} = Z + v\) of \(Z\) such that \(Z' \subset B_{\delta_0}(x, c\delta_0) \subset I\). Indeed one can take \(v\) to be the vector that translates \(l_0 \in \text{cl} Z\) to \(x\), the center of the ball \(B_{\delta_0}(x, \delta_0) \subset I\). Note that, by a similar argument to the proof of (7.5), \(\exists c_Z \in (0, 1)\) s.t. \(\mathbf{m}(I \setminus Z_{\delta_0}) \geq c_Z \mathbf{m}(I)\). Now since \(\mathbf{m}(I \setminus Z) \geq \mathbf{m}(I) - \mathbf{m}(Z) = \mathbf{m}(I) - \mathbf{m}(Z_{\delta_0}) = \mathbf{m}(I \setminus Z_{\delta_0})\), it follows that \(\mathbf{m}(I \setminus Z) \geq c_Z \mathbf{m}(I)\) hence (7.8) is satisfied.

The proof of (7.9) is similar to the proof of (7.6).

As for (7.10), we can show the stronger statement that \(TZ \supset Z\). Indeed, since \(Z\) is a rectangle (of positive side-lengths) bordering \(L\) and partition elements of \(T\) accumulate on \(L\), it is easy to see that there exists \(i_0 \in \mathbb{N}\) such that \(\forall i \geq i_0\) there exists \(1 \leq j = j(i) \leq 5^i\) such that \(Z\) contains the sets \(O_{i,j+k}\) for \(k \in \{1, 2, \ldots, 5\}\). In particular, if we let \(j_0 := j(i_0)\) and \(Q_1 = \{O_{i_0,j_0+1}, \ldots, O_{i_0,j_0+5}\}\), then \(TZ \supset \bigcup_{Q \in Q_1} TO\). Since \(\bigcup_{Q \in Q_1} TO = X\) in our example, it follows that \(TZ \supset Z\) hence (7.10) is satisfied.

It remains to finish the proof of item (3) of (8). Note that any finite sub-collection of \(\bigcup_{i \geq i_0} \{O_{i,j(i)+k}\}_{k=1}^{5}\) meets the requirements for \(\mathcal{P}_Z\) and the proof of (7.15) is again similar to the proof of (7.6).

\[\square\]

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