SINGULAR LIMITS OF ABSOLUTELY CONTINUOUS INVARIANT MEASURES FOR FAMILIES OF TRANSITIVE MAPS

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Abstract. We investigate the dependence on the parameters of absolutely continuous invariant measures for a family of piecewise linear piecewise expanding maps. We construct an example to show that the transitivity of the maps does not imply the convergence of those measures to the absolutely continuous invariant measure for the limit map.

1. Introduction

The existence of chaos in deterministic systems has been known for a long time. In such systems it is impossible to make accurate predictions of the long-term behavior of trajectories. However, it may be possible to make statistical predictions. This is due to the existence of invariant measures. Of such measures, the ones that are absolutely continuous with respect to the Lebesgue measure play the most important role. In particular, these measures are physically meaningful. Consider a system with a unique absolutely continuous invariant measure (*acim* in short). In practice, due to measurement errors, one is really dealing with a perturbation of the system. It is natural to ask whether the acim of the perturbed system is in some sense close to the acim of the unperturbed system – this is an important kind of stability. We will consider here one-dimensional dynamical systems and show that even in very simple systems the question of this type of stability is not trivial.

When we deal with piecewise expanding maps, we know that for each of them an acim exists, as was proved by Lasota and Yorke [4]. Moreover, if the map is transitive, then this measure is unique (it follows immediately from the results of [5]). Consider the case when there is an invariant interval such that the trajectory of almost every point falls into this interval, and the map restricted to this interval is transitive. Then there is also a unique acim, and it is supported by this invariant interval. Keller in [3] used this property to construct an example in which such an interval exists for some interval of parameters, and as the parameter converges to a limit value, those intervals become shorter and shorter. Then the weak-$*$ limit of acims is a measure concentrated at one point, while the limit map is transitive and has an acim with the support equal to the whole phase space. He conjectured that this is the only mechanism in which the continuity of the acims can be violated. We are showing here that other mechanisms can exist.

The paper is organized as follows. In Section 2 we describe quickly Keller's example. In Section 3 we construct our own example. Then we study it in Section 4, where we compute the invariant density, and in Section 5, where we compute limit measures. In Section 6 we look what happens if the slopes on laps (intervals of monotonicity) are constant, similarly

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as in the Keller's example. Finally, in Section 7 we take another look at what we did, and pose some questions.

2. Keller's example

Keller [3] showed that a large class of piecewise expanding maps, namely those that admit uniform Lasota-Yorke bounds, are acim-stable (in the sense of weak-$*$ convergence of acims). However, many simple dynamical systems exist that do not fall into this category. Keller's example mentioned in the preceding section looks as follows. Consider a 3-parameter family $\{W_{a,b,r} : 1/2 \le a, b \le 1;\ 0 < r < 1/2\}$ of maps of the interval $[0,1]$ to itself, defined on $[0,1/2]$ by

$$W_{a,b,r}(x) = \begin{cases} a\left(1 - \frac{x}{r}\right) & \text{if } 0 \le x \le r, \\ \frac{2b}{1-2r}(x - r) & \text{if } r \le x \le 1/2, \end{cases}$$

and on $(1/2, 1]$ by $W_{a,b,r}(x) = W_{a,b,r}(1 - x)$ (see Figure 1). Those maps are piecewise

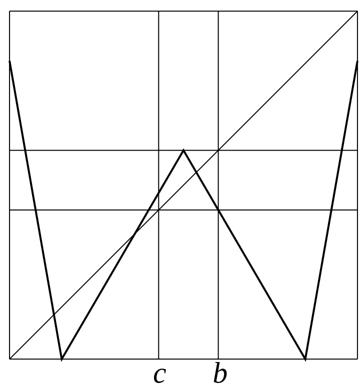

Figure 1. Map $W_{6/7,3/5,3/20}$.

expanding and if $1/2 < b \le 1 - 2r$ then the trajectory of almost every point falls into the invariant interval $[c, b]$, where $c = 2b(1 - b - r)/(1 - 2r)$, on which the map is transitive. Thus, for any sequence (a_n, b_n, r_n) converging to $(a, 1/2, 1/4)$, if $1/2 < b_n \le 1 - 2r_n$ for all n, the acims of W_{a_n,b_n,r_n} converge to the measure concentrated at $1/2$. On the other hand, $W_{a,1/2,1/4}([0, 1]) = [0, a]$ and on $[0, a]$ this map is transitive. Therefore it has an acim with the support $[0, a]$. Keller conjectured that for continuous maps of the interval, the only way such acim-instability can occur is if small neighborhoods of the orbit of a periodic turning point of the unperturbed map are invariant under the perturbed maps.

The acim-instability of a dynamical system is closely related to sensitive dependence on parameters defined by the second author in [6]. It is shown in [6] that the popular class of logistic maps has sensitive dependence on parameters which implies they are not acim-stable. However, there the acim-instability is based on the fact that for most of the maps there is no acim, and instead we consider Sinai-Ruelle-Bowen (or *physical*) measures, that are often concentrated on attracting periodic orbits.

Let us also mention that by the result of Raith [7], if the family of the maps consists of unimodal maps with the constant slope, in this family we have acim-stability (when we say "slope," we mean the absolute value of the slope).

3. Construction of the example

Let $I = [0,1]$ and let $T : I \to I$ be a continuous map. Let $\mathcal{P}$ be a partition of I given by the points $0 = a_0 < a_1 < \cdots < a_n = 1$. For $i = 1, \ldots, n$ let $I_i = [a_{i-1}, a_i]$ and denote the restriction of T to I_i by T_i. If T_i is a homeomorphism from I_i onto some connected union of intervals of $\mathcal{P}$, i.e., some interval $[a_{j(i)}, a_{k(i)}]$, then T is said to be Markov. The partition $\mathcal{P} = \{I_i\}_{i=1}^n$ is referred to as a Markov partition with respect to T. If each T_i is also linear on I_i, we say T is a piecewise linear Markov transformation. For a piecewise linear Markov transformation we define the incidence matrix $A_T = (a_{ij})_{i,j=1}^n$ induced by T and $\mathcal{P}$ by

$$a_{ij} = \begin{cases} 1 & \text{if } I_j \subset T(I_i), \\ 0 & \text{otherwise.} \end{cases}$$

In Keller's example perturbations of the map left a small neighborhood of $1/2$ (the turning fixed point of the unperturbed map) invariant. Therefore the measure piled up at $1/2$ as the size of the perturbation decreased. In our example we allow for the leak of the measure from small neighborhoods of $1/2$. We have some nearly invariant interval surrounding $1/2$; we will call it the *box*, because we are thinking about the graph of the map. We define perturbations of the map such that the measure can escape through some small interval centered at $1/2$, contained in the box. If Keller's conjecture were true, we would see a convergence of acims of perturbed maps to the acim of the limiting map. However, by controlling how fast the measure escapes out of the box and how fast it comes back into it, we prove that the measure can still pile up at $1/2$. We define perturbations based on three parameters a, b and c, as shown in Figure 2. Parameter a represents the size of the box. Parameter b is the size of the opening through which measure escapes. In this way we control how much of the measure escapes out of the box. Parameter c is the height of the peak that sticks out of the box; it controls how long the measure stays out of the box.

More precisely, we define a 3-parameter family $W(a,b,c)$ of piecewise linear maps of the unit interval as follows. If $0 \le x \le 1/2$ then

$$W(a,b,c)(x) = \begin{cases} (1-4x) & \text{if } 0 \le x \le \frac{1}{4}, \\ \frac{2-2a}{1-2a}\left(x - \frac{1}{4}\right) & \text{if } \frac{1}{4} \le x \le \frac{1-a}{2}, \\ \frac{2a}{a-b}\, x - \frac{(1-a)(a+b)}{2(a-b)} & \text{if } \frac{1-a}{2} \le x \le \frac{1-b}{2}, \\ \frac{2c}{b}\, x + \frac{1+a}{2} + c - \frac{c}{b} & \text{if } \frac{1-b}{2} \le x \le \frac{1}{2}; \end{cases}$$

and if $1/2 < x \le 1$ then $W(a,b,c)(x) = W(a,b,c)(1-x)$ (see Figure 2). In particular, we have $W(a,b,c)(0) = 1$, $W(a,b,c)(1/4) = 0$, $W(a,b,c)((1-a)/2) = (1-a)/2$, $W(a,b,c)((1-b)/2) = (1+a)/2$, $W(a,b,c)(1/2) = (1+a)/2 + c$. Let

$$s_1 = 4, \quad s_2 = \frac{2-2a}{1-2a}, \quad s_3 = \frac{2a}{a-b}, \quad s_4 = \frac{2c}{b} \tag{3.1}$$

denote the slopes of $W(a,b,c)$ on the consecutive pieces of $[0,1/2]$ on which the slope is constant.

Lemma 3.1. *If $0 < b < a < 1/2$ and $b < c \le (1-a)/2$, then the map $W(a,b,c)$ is transitive. Likewise, the map $W(0,0,0)$ is transitive.*

Proof. Assume that $0 < b < a < 1/2$ and $b < c \le (1-a)/2$. Then all the slopes are larger than some constant $\alpha > 2$. Suppose an interval J of length $|J|$ is contained in a lap

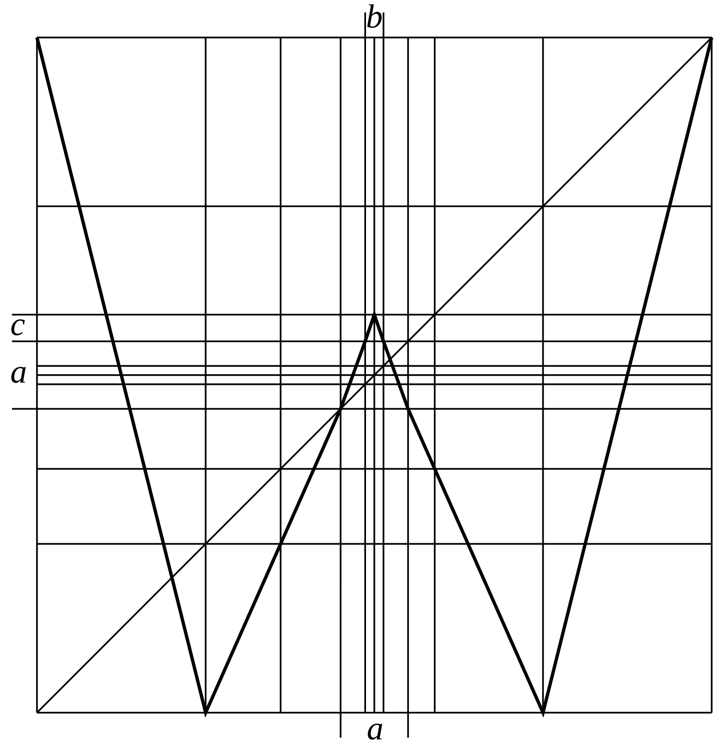

FIGURE 2. Graph of $W(a,b,c) = W_2(a,b)$ for $a = 1/10$, $b = 11/405$, $c = 16/405$, and its Markov partition.

of $W(a,b,c)$. Then $W(a,b,c)(J)$ either contains a lap, or contains an interval K contained in a lap, with $|K| > (\alpha/2)|J|$. Since $\alpha/2 > 1$, this proves that for some n the interval $W^n(a,b,c)(J)$ contains a lap. Then $W^{n+1}(a,b,c)(J) \supset [0,1/2]$, and $W^{n+2}(a,b,c)(J) = [0,1]$. This proves transitivity of $W(a,b,c)$.

For $W(0,0,0)$ the situation is a little more complicated, because the slopes of the second and third laps are equal to 2. However, if $K = W(0,0,0)(J)$ is contained in the union of the first and second laps or in the union of the third and fourth laps, then (because the slope of the first and fourth laps is 4) the length of $W^2(0,0,0)(J)$ is equal to $\max(4p, 2q)$ for some non-negative p, q with $p + q = |K|$. The function $p \mapsto 4p$ is increasing, while the function $p \mapsto 2(|K| - p)$ is decreasing. Therefore the minimum of $\max(4p, 2q)$ occurs at the point where $4p = 2q$, that is, $p = (1/3)|K|$. This proves that $\max(4p, 2q) \geq (4/3)|K|$. Thus, the only reason why the proof from the preceding paragraph may not work for $W(0,0,0)$ is that $W^k(0,0,0)(J)$ contains $1/2$ in its interior for some k. However, $1/2$ is a fixed point, and its left-hand-sided neighborhood grows under the action of $W(0,0,0)$ until some image contains the second lap. Then the next image contains the interval $[0,1/2]$, and again we get transitivity. $\square$

We will show that there exists a sequence (a_n, b_n, c_n) converging to $(0,0,0)$ such that the unique acims of $W(a_n, b_n, c_n)$ converge to the measure concentrated at $1/2$ instead of the acim of $W(0,0,0)$. With other choices of (a_n, b_n, c_n), other behaviors are possible, as described in Theorem 5.1. We will choose the sequence (a_n, b_n, c_n) so that the maps $W(a_n, b_n, c_n)$ are Markov. More precisely, we require that $1/2$ is mapped to a point on the third lap, then for some time the trajectory stays on the second lap, being repelled from the fixed point $(1-a)/2$, until it gets to $1/4$. The number n is such that $W^{n+1}(a_n, b_n, c_n)(1/2) = 1/4$. The point symmetric to $W(a,b,c)(1/2)$ with respect to $1/2$ is $(1-a)/2 - c$ and the slope on the interval $[1/4, (1-a)/2]$ is s_2. Thus we get the equation

$$c \cdot \left(\frac{2-2a}{1-2a}\right)^n = \frac{1-a}{2} - \frac{1}{4}.$$

The solution to this equation is

$$c = c_n(a) = \frac{1-2a}{4}\left(\frac{1-2a}{2-2a}\right)^n. \tag{3.2}$$

When we specify a_n and b_n, then we will take $c_n = c_n(a_n)$.

Let us denote $W_n(a,b) = W(a,b,c_n(a))$. This map is a Markov map on $n+8$ subintervals $\{I_i\}_{i=1}^{n+8}$. The first subinterval is $[0,1/4]$. then there come n subintervals of $[1/4,(1-a)/2]$ determined by the images of $1/2$, then 4 subintervals of the box, 2 subintervals of $[(1+a)/2, 3/4]$, and finally $[3/4,1]$ (see Figure 2).

4. Invariant density

In a general case, the density of an acim for a map of the interval cannot be written in a closed form. However, for a piecewise linear Markov map this density can be calculated. Let T be a piecewise linear Markov map with incidence matrix $A_T = (a_{ij})_{i,j=1}^k$. Define $M_T = (m_{ij})_{i,j=1}^k$ by $m_{ij} = a_{ij}/|T_i'|$. If T admits a unique invariant density, then the invariant density is piecewise constant on the intervals of the Markov partition and is given by the left eigenvector of the matrix M_T corresponding to eigenvalue 1 (for a reference see [2]). This vector is normalized so that the total measure is 1.

Let A_n be the incidence matrix for $W_n(a,b)$. Then the entry a_{ij} of A_n is equal to 1 in the following cases:

- $1 \le j \le n$ and $i \in \{1, j+1, n+7, n+8\}$,
- $j = n+1$ and $i \in \{1, n+1, n+6, n+8\}$,
- $n+2 \le j \le n+5$ and $i \in \{1, n+2, n+5, n+8\}$,
- $j = n+6$ and $i \in \{1, n+3, n+4, n+8\}$,
- $n+7 \le j \le n+8$ and $i \in \{1, n+8\}$.

The slopes $|T_i'|$, according to (3.1), are

- 4 if $i \in \{1, n+8\}$,
- $(2-2a)/(1-2a)$ if $2 \le i \le n+1$ or $i \in \{n+6, n+7\}$,
- $2a/(a-b)$ if $i \in \{n+2, n+5\}$,
- $2c_n(a)/b$ if $i \in \{n+3, n+4\}$.

This gives us the following equations for our eigenvector $(x_1, x_2, \ldots, x_{n+8})$. If $1 \le j \le n$ then

$$x_j = \frac{x_1}{4} + \frac{1-2a}{2-2a}x_{j+1} + \frac{1-2a}{2-2a}x_{n+7} + \frac{x_{n+8}}{4}, \tag{4.1}$$

$$x_{n+1} = \frac{x_1}{4} + \frac{1-2a}{2-2a}x_{n+1} + \frac{1-2a}{2-2a}x_{n+6} + \frac{x_{n+8}}{4}, \tag{4.2}$$

if $n+2 \le j \le n+5$ then

$$x_j = \frac{x_1}{4} + \frac{a-b}{2a}x_{n+2} + \frac{a-b}{2a}x_{n+5} + \frac{x_{n+8}}{4}, \tag{4.3}$$

$$x_{n+6} = \frac{x_1}{4} + \frac{b}{2c_n(a)}x_{n+3} + \frac{b}{2c_n(a)}x_{n+4} + \frac{x_{n+8}}{4}, \tag{4.4}$$

if $n+7 \le j \le n+8$ then

$$x_j = \frac{x_1}{4} + \frac{x_{n+8}}{4}. \tag{4.5}$$

Set

$$x_1 = 1. \tag{4.6}$$

Then from (4.5) we get

$$x_{n+7} = x_{n+8} = \frac{1}{3}. \tag{4.7}$$

Next, from (4.3) we get

$$x_{n+2} = x_{n+3} = x_{n+4} = x_{n+5} = \frac{a}{3b}. \tag{4.8}$$

Further, from (4.4) we get

$$x_{n+6} = \frac{1}{3} + \frac{a}{3c_n(a)}. \tag{4.9}$$

Finally, from (4.2) we get

$$x_{n+1} = 1 - \frac{4a}{3} + \frac{a(1-2a)}{3c_n(a)}. \tag{4.10}$$

Now, in order to compute $x_2, x_3, \dots, x_n$, we rewrite (4.1) as

$$x_{j+1} = \frac{2-2a}{1-2a}x_j - \frac{3-4a}{3-6a}.$$

From this, we get by induction

$$x_j = 1 - \frac{4a}{3} + \frac{4a}{3}\left(\frac{2-2a}{1-2a}\right)^{j-1}.$$

Taking into account (3.2), we get for $1 \le j \le n+1$

$$x_j = 1 - \frac{4a}{3} + \frac{a(1-2a)}{3c_{j-1}(a)}. \tag{4.11}$$

Note that for $j = n+1$ this agrees with (4.10).

Now we have to find the normalizing factor

$$C = \sum_{j=1}^{n+8} |I_j| x_j. \tag{4.12}$$

The lengths of intervals I_j of our Markov partition are:

- $1/4$ if $j \in \{1, n+8\}$,
- $c_{j-2} - c_{j-1}$ if $2 \le j \le n$,
- c_{n-1} if $j = n+1$,
- $(a-b)/2$ if $j \in \{n+2, n+5\}$,
- $b/2$ if $j \in \{n+3, n+4\}$,
- $c_n(a)$ if $j = n+6$,
- $(1-2a)/4 - c_n(a)$ if $j = n+7$.

Let us look at various parts of the sum (4.12) and their limits as a, b go to 0 and n goes to infinity (so $c_n(a) \to 0$). We have

$$|I_1|x_1 + \sum_{j=n+6}^{n+8} |I_j|x_j = \frac{1}{4}\cdot 1 + c_n(a)\cdot\left(\frac{1}{3} + \frac{a}{3c_n(a)}\right) + \left(\frac{1-2a}{4} - c_n(a)\right)\cdot\frac{1}{3} + \frac{1}{4}\cdot\frac{1}{3} \to \frac{5}{12}, \tag{4.13}$$

$$\sum_{j=n+2}^{n+5} |I_j|x_j = \left(2\cdot\frac{a-b}{2} + 2\cdot\frac{b}{2}\right)\cdot\frac{a}{3b} = \frac{a^2}{3b}, \tag{4.14}$$

and

$$|I_{n+1}|x_{n+1} = c_{n-1}(a)\cdot\left(1 - \frac{4a}{3} + \frac{a(1-2a)}{3c_n(a)}\right) \to 0. \tag{4.15}$$

Finally,

$$\sum_{j=2}^{n} |I_j|x_j = \sum_{j=2}^{n}(c_{j-2}(a) - c_{j-1}(a))\cdot\left(1 - \frac{4a}{3} + \frac{a(1-2a)}{3c_{j-1}(a)}\right).$$

We will compute this sum in two steps. First,

$$\sum_{j=2}^{n}(c_{j-2}(a) - c_{j-1}(a))\cdot\left(1 - \frac{4a}{3}\right) = (c_0(a) - c_{n-1}(a))\frac{3-4a}{3} \to \frac{1}{4}. \tag{4.16}$$

Next, since $c_{j-2}(a) - c_{j-1}(a) = c_{j-1}(a)/(1-2a)$, we have

$$\sum_{j=2}^{n}(c_{j-2}(a) - c_{j-1}(a))\cdot\frac{a(1-2a)}{3c_{j-1}(a)} = (n-1)\frac{a}{3}. \tag{4.17}$$

Thus, we get

$$\liminf \sum_{j=2}^{n} |I_j|x_j = \frac{1}{4} + \liminf \frac{na}{3}, \quad \limsup \sum_{j=2}^{n} |I_j|x_j = \frac{1}{4} + \limsup \frac{na}{3}. \tag{4.18}$$

Now we see that the behavior of the invariant density as a, b go to 0 and n goes to infinity depends on the behavior of the quantities a^2/b and na. However, it turns out that only a^2/b matters.

Lemma 4.1. *Let $0 < b_n < a_n < 1/2$ and $b_n < c_n \le (1-a_n)/2$ with $a_n, b_n, c_n \to 0$ as $n \to \infty$, and $c_n = c_n(a)$. If $na_n \to \alpha$ on a subsequence, with $\alpha \in (0, \infty]$, then $(a_n^2/b_n)/(na_n) \to \infty$ on the same subsequence.*

Proof. By (3.2), we have $c_n < 2^{-n-2}$, so $1/b_n > 2^{n+2}$. Therefore

$$\frac{a_n^2/b_n}{(na_n)^2} > \frac{2^{n+2}}{n^2} \to \infty$$

as $n \to \infty$. Thus, if $na_n \to \alpha > 0$ on a subsequence, then $(a_n^2/b_n)/(na_n) \to \alpha\cdot\infty = \infty$ on the same subsequence. □

Using the same methods, it is very easy to find the density of the acim for $W(0,0,0)$. We get a Markov partition into 4 intervals: $[0, 1/4]$, $[1/4, 1/2]$, $[1/2, 3/4]$ and $[3/4, 1]$. The density on the first two intervals is $3/2$, and on the last two $1/2$.

5. Limit measures

Now we investigate what happens with the acims μ_n for $W_n(a_n, b_n)$ as n goes to infinity and a_n, b_n go to 0. We denote by μ the acim for $W(0,0,0)$ and by $\delta_{1/2}$ the Dirac's delta measure at $1/2$.

Theorem 5.1. *Let $0 < b_n < a_n < 1/2$ and $b_n < c_n(a_n) \le (1-a_n)/2$ with $a_n, b_n \to 0$ and $a_n^2/b_n \to \beta \in [0,\infty]$ as $n \to \infty$. Then*

$$\lim_{n\to\infty} \mu_n = \frac{2}{2+\beta}\mu + \frac{\beta}{2+\beta}\delta_{1/2} \tag{5.1}$$

in the weak-$*$ topology.

Proof. We can write μ_n as the sum of three measures: $\nu_n + \sigma_n + \tau_n$, defined as follows. They are all absolutely continuous with respect to the Lebesgue measure, and their densities are:

- for ν_n:
  x_j/C on I_j for $j = 1, n+1, n+6, n+7, n+8$,
  $(3-4a)/(3C)$ on I_j for $j = 2, \dots, n$,
  0 on I_j for $j = n+2, \dots, n+5$,
- for σ_n:
  x_j/C on I_j for $j = n+2, \dots, n+5$,
  0 on all other I_j,
- for τ_n:
  $a(1-2a)/(3Cc_{j-1}(a))$ on I_j for $j = 2, \dots, n$,
  0 on all other I_j,

where $a = a_n$, and I_j, x_j and C depend on n.

Consider now three cases, depending on the value of β.

Case I: $\beta = 0$. Then by (4.14), $\sigma_n \to 0$. Moreover, by Lemma 4.1, $na_n \to 0$, so by (4.17) $\tau_n \to 0$. Therefore the limit of the measures μ_n is the same as the limit of measures ν_n. By (4.13), (4.14), (4.15) and (4.18), the limit of C as $n \to \infty$ is $5/12 + 1/4 = 2/3$, and thus the density of ν_n is $3/2$ on $[0,1/4]$, $1/2$ on $I_{n+7} \cup I_{n+8}$ (and this interval converges to $[1/2,1]$), and $(3-4a)/2$ on $\bigcup_{j=2}^n I_j$ (and this interval converges to $[1/4,1/2]$. The total measure on remaining intervals converges to 0, and thus $\nu_n \to \mu$. This proves (5.1) in this case.

Case II: $\beta \in (0,\infty)$. The only difference between this case and the preceding one is that this time σ_n converges to a positive constant times $\delta_{1/2}$. This changes the constant by which we divide x_j's to get the density of ν_n. By (4.14) and the computations from Case I we know that

$$\lim_{n\to\infty} \sigma_n([0,1]) / \lim_{n\to\infty} \nu_n([0,1]) = \frac{\beta/3}{2/3} = \frac{\beta}{2}.$$

Thus the constant mentioned above is $\beta/(2+\beta)$, and the limit of the measures ν_n is $2/(2+\beta)$ times μ instead of just μ. This proves (5.1) in this case.

Case III: $\beta = \infty$. Then

$$\lim_{n\to\infty} \sigma_n([0,1]) / \lim_{n\to\infty} \nu_n([0,1]) = \lim_{n\to\infty} \sigma_n([0,1]) / \lim_{n\to\infty} \tau_n([0,1]) = \infty,$$

so

$$\lim_{n\to\infty} \mu_n = \lim_{n\to\infty} \sigma_n = \delta_{1/2}.$$

This proves (5.1) in this case. ∎

The above theorem does not yet prove that the example we claimed we built really exists. Namely, we have to show that the sequences (a_n) and (b_n) satisfying its conditions exist. We can also settle the question whether in such examples we can have slopes bounded independently of n.

Theorem 5.2. *For every $\beta \in [0,\infty]$ there exist sequences (a_n) and (b_n) satisfying the assumptions of Theorem 5.1 and such that for sufficiently large n all slopes of the maps $W_n(a_n,b_n)$ are in $(2,4]$.*

Proof. The slopes of $W_n(a_n,b_n)$ are 4, $(2-2a_n)/(1-2a_n)$, $2a_n/(a_n-b_n)$ and $2c_n(a_n)/b_n$. Under the assumptions of Theorem 5.1, they are all larger than 2. Additional conditions guaranteeing that they are not larger than 4 are

$$a_n \le \frac{1}{3}, \quad b_n \le \frac{a_n}{2}, \quad c_n(a_n) \le 2b_n.$$

Thus, we need to show that we can find sequences (a_n) and (b_n) of positive numbers convergent to 0, with $a_n^2/b_n \to \beta$ and

$$2b_n \le a_n \le \frac{1}{3}, \quad \frac{c_n(a_n)}{2} \le b_n < c_n(a_n) < \frac{1-a_n}{2}$$

when n is sufficiently large.

We define numbers β_n as follows. If $\beta = 0$ then $\beta_n = 1/n$. If $\beta \in (0,\infty)$ then $\beta_n = \beta$ for all n. If $\beta = \infty$, then $\beta_n = n$. Then we define continuous functions $f_n : [0,1/2) \to \mathbb{R}$ by

$$f_n(a) = \frac{5a^2}{4c_n(a)}.$$

Note that $f_n(0) = 0$ and if $a > 0$ then $f_n(a) > 5a^2 2^n$. For all values of β we have $\sqrt{\beta_n/5} \cdot 2^{-n/2} \to 0$, so for sufficiently large n there exists $a_n \in (0, \sqrt{\beta_n/5} \cdot 2^{-n/2})$ such that $f_n(a_n) = \beta_n$, and we have $a_n \to 0$. Therefore $c_n(a_n) < (1-a_n)/2$ for sufficiently large n.

Set $b_n = (4/5)c_n(a_n)$. Then $c_n(a_n)/2 < b_n < c_n(a_n)$. Moreover, $a_n^2/b_n = f_n(a_n) = \beta_n$, so $a_n^2/b_n \to \beta$. We have

$$\frac{a_n}{b_n} = \frac{\beta_n}{a_n} > \frac{\beta_n}{\sqrt{\beta_n/5} \cdot 2^{-n/2}} = \sqrt{5\beta_n} \cdot 2^{n/2} \to \infty.$$

Therefore $2b_n \le a_n$ for sufficiently large n. Thus, the sequences (a_n) and (b_n) satisfy all properties they were supposed to satisfy. □

6. Maps with constant slopes on laps

In this section we study the special case when the slope of $W(a,b,c)$ is constant on each lap of the map (see Figure 3). This means that

$$\frac{2-2a}{1-2a} = \frac{2a}{a-b} = \frac{2c_n(a)}{b},$$

that is,

$$b = \frac{a^2}{1-a}, \quad c_n(a) = \frac{a^2}{1-2a}. \tag{6.1}$$

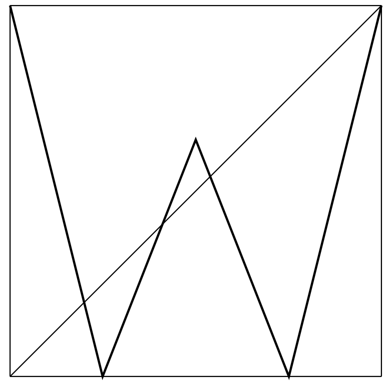

FIGURE 3. Map $W(a,b,c)$ with constant slopes on laps.

In view of (3.2), we get an equation

$$\left(\frac{2a}{1-2a}\right)^2 = \left(\frac{1-2a}{2-2a}\right)^n.$$

For $a = 0$, the left hand side of this equation is 0, while the right hand side is positive. For $a = 2^{-(n+2)/2}$ the left hand side is larger than 2^{-n}, while the right hand side is smaller than 2^{-n}. Therefore it has a solution $a_n \in (0, 2^{-(n+2)/2})$. Then we set $b_n = a_n^2/(1-a_n)$.

Let us check whether the assumptions of Theorem 5.1 are satisfied. Since $a_n \in (0, 2^{-(n+2)/2})$, we get $a_n < 1/2$ and $a_n \to 0$. Then $0 < b_n < a_n$ and $b_n < c_n(a_n)$ follow immediately from (6.1). The inequality $c_n(a_n) \le (1-a_n)/2$ is equivalent to $a_n \le 1/3$, so it is satisfied for all $n \ge 2$. We have $a_n^2/b_n = 1 - a_n \to 1$. Therefore, by Theorem 5.1 we get

$$\mu_n \to \frac{2}{3}\mu + \frac{1}{3}\delta_{1/2}.$$

Thus, even in this simple case the limit of the acims of the maps $W_n(a_n, b_n)$ is not the acim for $W(0,0,0)$.

7. Discussion and questions

Let us look again at our example. As we mentioned in Section 3, parameters a, b and c play different roles. The size b of the hole in the box, compared to the size a of the box, determine how fast the measure leaks from the box. The parameter c controls how long the part of the measure that left the box stays outside. However, according to Theorem 5.1, only the ratio a^2/b plays any role in determining the limit behavior. This is due to the additional assumption that $b < c$. It is a technical assumption, used in Lemma 3.1 to make the slopes larger than 2. In fact, that lemma is probably also true without this assumption; while some slopes may be even less than 1, for an appropriate iterate of the map they should become larger than 2. Thus, we are left with the question: why does it seem that the size of c is irrelevant in the limit behavior of acims? The answer is in Lemma 4.1. For this lemma to hold, we need $n^2 b$ to converge to 0, and if c is too small then n is too large. Thus, the heuristic arguments are correct.

Let us now pose a couple of questions. The first one is whether it is important in our example that the maps are Markov (or even Markov with this specific Markov partition).

While the "common sense" suggests that everything should be similar in the non-Markov case, estimates of the density of the acim do not seem to be simple.

The second question is about unimodal maps. As we mentioned in Section 2, if the family of the maps consists of unimodal maps with the constant slope, in this family we have acim-stability. However, there is an interesting family of unimodal maps, for which the acim-stability is unknown. It is defined as follows (see Figure 4).

$$A(a,b)(x) = \begin{cases} \frac{1-a}{b}\,x + a & \text{if } 0 \le x \le b, \\ \frac{1}{1-b}(1-x) & \text{if } b \le x \le 1. \end{cases}$$

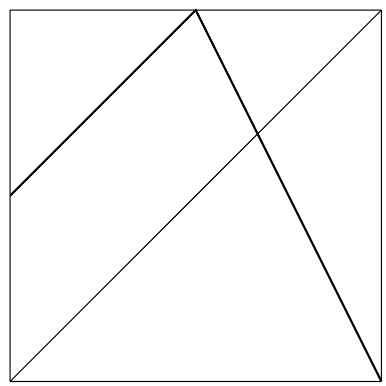

FIGURE 4. Map $A(1/2, 1/2)$.

Consider the map $A(1/2, 1/2)$. It seems that this map is acim-stable in this family. This example is the simplest example one can make whose acim-stability seem not to follow from any of the existing techniques. We remark that this map is not a "good" map as defined in [1]. A unimodal map is good in this sense if its critical point is not periodic or it is periodic of period n and $\inf |(f^n)'| > 2$.

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REFERENCES

[1] V. Baladi and D. Smania, Linear response formula for piecewise expanding unimodal maps, *Nonlinearity* **21** (2008) 677–711.

[2] A. Boyarsky and P. Góra, *Laws of Chaos. Invariant Measures and Dynamical Systems in One Dimension*, Probability and its Applications, Birkhäuser, Boston, MA, 1997.

[3] G. Keller, Stochastic stability in some chaotic dynamical systems, *Monatshefte für Mathematik* **94** (1982) 313–333.

[4] A. Lasota and J. A. Yorke, On the existence of invariant measures for piecewise monotonic transformations, *Trans. Amer. Math. Soc.* **186** (1973), 481–488.

[5] T.-Y. Li and J. A. Yorke, Ergodic transformations from an interval into itself, *Trans. Amer. Math. Soc.* **235** (1978), 183–192.

[6] M. Misiurewicz, Sensitive dependence on parameters, *J. Math. Phys.* **51** (2010), 102704 (12 pages).

[7] P. Raith, Continuity of the measure of maximal entropy for unimodal maps on the interval, *Qual. Theory Dyn. Syst.* **4** (2003), 67–76.

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