PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 00, Number 0, Pages 000–000
S 0002-9939(XX)0000-0

STRONGER LASOTA-YORKE INEQUALITY FOR ONE-DIMENSIONAL PIECEWISE EXPANDING TRANSFORMATIONS

PEYMAN ESLAMI AND PAWEL GÓRA

ABSTRACT. For a large class of piecewise expanding $\mathcal{C}^{1,1}$ maps of the interval we prove the Lasota-Yorke inequality with a constant smaller than the previously known $2/\inf|\tau'|$. Consequently, the stability results of Keller-Liverani [7] apply to this class and in particular to maps with periodic turning points. One of the applications is the stability of acim's for a class of W-shaped maps. Another application is an affirmative answer to a conjecture of Eslami-Misiurewicz [2] regarding acim-stability of a family of unimodal maps.

1. Introduction

The problem of stability in general and the stability of invariant measures in particular are one of the most important (and difficult) questions in dynamical systems. Here, we are concerned with the stability of absolutely continuous invariant measures (acim-stability) for piecewise expanding maps of an interval. The general setting is as follows.

Definition 1.1. (acim-stability) Given a family of maps $\{\tau_\epsilon : X \to X\}_{\epsilon \geq 0}$ with corresponding invariant densities $\{f_\epsilon\}_{\epsilon \geq 0}$, we say that τ_0 is acim-stable if $\lim_{\epsilon\to 0} \tau_\epsilon = \tau_0$ implies $\lim_{\epsilon \to 0} f_\epsilon = f_0$; the limits are taken with respect to properly chosen metrics on the space of maps and densities, respectively.

A relevant notion of closeness for maps under consideration is convergence in the Skorokhod metric (see Definition 4.4), and for the corresponding invariant densities, in this paper, is convergence in $\mathcal{L}^1$.

Stability problems were investigated in a multitude of works; most relevant to our study are [6] and [7].

The main motivation for this work was to prove acim-stability for some W-shaped maps with slopes > 1 (by slope we shall always mean absolute value of the slope). A troublesome property of such maps is that they contain periodic turning points. Let us consider such a map W with a fixed turning point p_0. This would not be a problem if $|W'| > 2$ (whenever the derivative exists). In fact, then the acim-stability of W follows directly from the results of [6]. However, if $1 < |W'| \leq 2$ near p_0, the standard procedure, which is to work with an iterate of W that has derivative > 2, fails due to the presence of the fixed turning point p_0. We bypass this problem by deriving a stronger Lasota-Yorke inequality, hence avoiding the iteration of the maps.

Received by the editors October 6, 2011.
2010 *Mathematics Subject Classification.* Primary 37A10, 37A05, 37E05.

A. Lasota and J. Yorke [10] first discovered this inequality and used it to prove the existence of acim's for piecewise expanding $\mathcal{C}^2$ maps. Z. Kowalski [8] later proved the existence of acim's for piecewise expanding $\mathcal{C}^{1,1}$ maps.

In this work we consider piecewise expanding $\mathcal{C}^{1,1}$ maps of an interval (see [9] for higher-dimensional results[1]). We prove a Lasota-Yorke inequality with a constant which is smaller than the previously known $2/\inf|\tau'|$, for a fairly large class of maps. This allows us to apply the stability theorems of [7]. One of the implications would be the acim-stability of a class of maps in the presence of periodic turning points.

We point out that perhaps one may be able to enlarge the class of maps under consideration to piecewise expanding $\mathcal{C}^{1+\epsilon}$ maps (i.e. with Hölder condition on derivatives); however, there are examples of piecewise expanding $\mathcal{C}^1$ maps with no acim's, as shown in [5].

2. Setting and notation

Suppose $I = [0,1]$ and m is the Lebesgue measure on I. We will be concerned with piecewise expanding $\mathcal{C}^{1,1}$ maps on I, which are defined as follows.

Definition 2.1 (piecewise expanding $\mathcal{C}^{1,1}$ functions)**.** Suppose there exists a partition $\mathcal{P} = \{I_i := (a_{i-1}, a_i),\ i = 1, \dots, q\}$ of I such that $\tau\colon I \to I$ satisfies the following conditions. For all i,

(1) $\tau_i := \tau_{|_{I_i}}$ is monotonic, $\mathcal{C}^1$, and can be extended to the closed interval $[a_{i-1}, a_i]$ as a $\mathcal{C}^1$ function;
(2) τ_i' is Lipschitz, i.e., there exists a constant M_i such that $|\tau_i'(x) - \tau_i'(y)| \le M_i|x-y|$, for all $x, y \in I_i$;
(3) $|\tau_i'(x)| \ge s_i > 1$ for all $x \in I_i$.

Then, we say $\tau \in \mathcal{T}(I)$, the class of piecewise expanding $\mathcal{C}^{1,1}$ maps on I.

If a family of maps $\{\tau_\epsilon\}$ satisfies the above conditions with uniform constants s_i and M_i (i.e. independent of ϵ), then we shall write $\{\tau_\epsilon\} \subset \mathcal{T}(I)$ uniformly.

We will use the following notation throughout the paper.

Let

$$s := \min_{1\le i\le q} s_i \quad \text{and} \quad M := \max_{1\le i\le q} M_i.$$

Also, let

$$\delta_i^{\pm} := \delta_{\{\tau(a_i^{\pm}) \notin \{0,1\}\}} = \begin{cases} 0 & \text{if } \tau(a_i^{\pm}) \in \{0,1\}, \\ 1 & \text{if } \tau(a_i^{\pm}) \notin \{0,1\}, \end{cases}$$

where $\tau(a_i^{\pm})$ means $\lim_{x\to a_i^{\pm}} \tau(x)$. For example, $\delta_i^+ = 1$ means that the left endpoint of the $(i+1)$-st branch of τ is hanging (doesn't touch 0 or 1).

We denote by P_τ the Perron-Frobenius operator induced by τ on $\mathcal{L}^1(I)$. That is

$$P_\tau f = \sum_{y\in\tau^{-1}(x)} g(y)f(y) = \sum_{i=1}^{q} \frac{f(\tau_i^{-1}(x))}{|\tau'(\tau_i^{-1}(x))|}\chi_{\tau(a_{i-1},a_i)}(x),$$

where $g(y) := 1/|\tau'(y)|$. Note that $\sup_{I_i}|g| \le 1/s_i < 1$.

[1] Upon a more detailed look at the approach of Liverani in [9], we believe that his approach, at least in the piecewise C^2 case, essentially leads to the same stronger Lasota-Yoke inequality which is presented in this paper.

The (total) variation of a function $f\colon I \to \mathbb{R}$ is defined as

$$\overline{\bigvee_I} f = \sup_{0=x_0<x_1<x_2<\cdots<x_N=1} \sum_{i=1}^{N} |f(x_i) - f(x_{i-1})|,$$

where the supremum is taken over all the partitions of the interval I.

The essential variation of a function $f\colon I \to \mathbb{R}$ is defined by:

$$\bigvee_I f = \inf_{g\simeq f} \overline{\bigvee_I} g,$$

where $\simeq$ denotes equality almost everywhere with respect to Lebesgue measure.

We will consider P_τ on the space of functions of bounded essential variation

$$BV(I) = \{f \in \mathcal{L}^1(I) : \bigvee_I f < \infty\}$$

modulo equality almost everywhere, with the norm

$$\|f\|_{BV} = \|f\|_{\mathcal{L}^1} + \bigvee_I f.$$

Since functions of bounded variation are continuous except at at most countable number of points at which they have one-sided limits, we assume that functions in $BV(I)$ satisfy

$$f(x_0) = \max\left\{\lim_{x\to x_0^-} f(x)\ ,\ \lim_{x\to x_0^+} f(x)\right\}\ , \tag{2.1}$$

at any point $x_0 \in I$. For such functions $\bigvee_I f = \overline{\bigvee}_I f$. For more information about $BV(I)$ we refer the reader to [1] and [3].

3. Lasota-Yorke inequality

Let

$$\eta_i := \begin{cases} \max\left\{\frac{\delta_0^+}{s_1}, \frac{\delta_1^+}{s_2}\right\} & \text{if } i = 1,\\ \max\left\{\frac{\delta_{q-1}^-}{s_{q-1}}, \frac{\delta_q^-}{s_q}\right\} & \text{if } i = q,\\ \max\left\{\frac{\delta_{i-1}^-}{s_{i-1}}, \frac{\delta_i^+}{s_{i+1}}\right\} & \text{for } i = 2 \dots q-1. \end{cases}$$

The new Lasota-Yorke inequality is given by the following.

Proposition 3.1 (New Lasota-Yorke inequality)**.** *Suppose $\tau \in \mathcal{T}(I)$, i.e., the class of piecewise expanding $\mathcal{C}^{1,1}$ maps on I. Then, for every $f \in BV(I)$,*

$$\bigvee_I P_\tau f \le \max_{1\le i\le q}\left\{\frac{1}{s_i} + \eta_i\right\}\bigvee_I f + \left[\frac{M}{s^2} + \frac{2\max_{1\le i\le q}\eta_i}{\min_{1\le i\le q} m(I_i)}\right]\int_I |f|dm. \tag{3.1}$$

Proof. We will estimate $\bigvee_I P_\tau f$. Let $0 = x_0 < x_1 < x_2 < \cdots < x_N = 1$ be a partition of I. We assume that the points $\tau_i(a_i^-)$ and $\tau_i(a_i^+)$, $i = 1, 2, \dots, q$, are included in this partition. This does not diminish the generality of the considerations. We also assume without loss of generality that

$$\max\{|x_j - x_{j-1}| : j = 1, 2, \dots, N\} \to 0 \text{ as } N \to \infty. \tag{3.2}$$

Let $I_i := (a_{i-1}, a_i)$. Let us use the following notation.

$$g_i(x) := g(\tau_i^{-1}(x))\chi_{\tau(I_i)}(x),\ f_i(x) := f(\tau_i^{-1}(x))\chi_{\tau(I_i)}(x).$$

Both functions, g_i and f_i, are supported on $\tau(I_i)$.

Let J_i denote the set of indices j such that x_{j-1} and $x_j \in \tau(I_i)$. We have

$$\sum_{j=1}^{N} |P_\tau f(x_j) - P_\tau f(x_{j-1})| \le \sum_{j=1}^{N}\sum_{i=1}^{q} |g_i(x_j)f_i(x_j) - g_i(x_{j-1})f_i(x_{j-1})|$$

$$\le \sum_{i=1}^{q}\sum_{j\in J_i} |g_i(x_j)f_i(x_j) - g_i(x_{j-1})f_i(x_{j-1})| + \sum_{i=1}^{q}\left(\left|g(a_{i-1}^+)f(a_{i-1}^+)\delta_{i-1}^+\right| + \left|g(a_i^-)f(a_i^-)\delta_i^-\right|\right)$$

$$\le \sum_{i=1}^{q}\sum_{j\in J_i} |f_i(x_j)\left(g_i(x_j) - g_i(x_{j-1})\right)| + \sum_{i=1}^{q}\sum_{j\in J_i} |g_i(x_{j-1})\left(f_i(x_j) - f_i(x_{j-1})\right)| + \sum_{i=1}^{q}\left(\frac{\delta_{i-1}^+}{s_i}|f(a_{i-1})| + \frac{\delta_i^-}{s_i}|f(a_i)|\right).$$

Using the Lipschitz condition on τ', we estimate the first sum above as follows.

$$\begin{aligned}\sum_{j\in J_i} |f_i(x_j)\left(g_i(x_j) - g_i(x_{j-1})\right)| &\le \sum_j \left| f_i(x_j)\frac{\tau'(\tau_i^{-1}(x_{j-1})) - \tau'(\tau_i^{-1}(x_j))}{\tau'(\tau_i^{-1}(x_j))\tau'(\tau_i^{-1}(x_{j-1}))}\right| \\ &\le \frac{M}{s^2}\sum_j \left|f(\tau_i^{-1}(x_j))\right|\left|\tau_i^{-1}(x_j) - \tau_i^{-1}(x_{j-1})\right| \\ &\le \frac{M}{s^2}\int_{I_i}|f|dm + \epsilon_i(N).\end{aligned}$$

The last sum above is a Riemann sum, which is estimated by the corresponding integral and an error term $\epsilon_i(N)$. It follows from assumption (3.2) that $\epsilon_i(N) \to 0$ as $N \to \infty$.

Using this estimate,

$$\sum_{j=1}^{N} |P_\tau f(x_j) - P_\tau f(x_{j-1})| \le \frac{M}{s^2}\sum_{i=1}^{q}\int_{I_i}|f| + \sum_{i=1}^{q}\epsilon_i(N) + \sum_{i=1}^{q}\frac{1}{s_i}\bigvee_{I_i} f + \sum_{i=1}^{q}\left(\frac{\delta_{i-1}^+}{s_i}|f(a_{i-1})| + \frac{\delta_i^-}{s_i}|f(a_i)|\right).$$

We divide the last sum into three groups and estimate as follows.

$$\begin{aligned}\frac{\delta_0^+}{s_1}|f(a_0)| + \frac{\delta_1^+}{s_2}|f(a_1)| &\le \max\left\{\frac{\delta_0^+}{s_1}, \frac{\delta_1^+}{s_2}\right\}\left(\bigvee_{I_1} f + 2\inf_{I_1}|f|\right) \\ &\le \max\left\{\frac{\delta_0^+}{s_1}, \frac{\delta_1^+}{s_2}\right\}\left(\bigvee_{I_1} f + \frac{2}{m(I_1)}\int_{I_1}|f|\right).\end{aligned}$$

Similarly,

$$\frac{\delta_{q-1}^-}{s_{q-1}}|f(a_{q-1})| + \frac{\delta_q^-}{s_q}|f(a_q)| \le \max\left\{\frac{\delta_{q-1}^-}{s_{q-1}}, \frac{\delta_q^-}{s_q}\right\}\left(\bigvee_{I_q} f + \frac{2}{m(I_q)}\int_{I_q}|f|\right).$$

Finally, for $i = 2, \ldots, q-1$,

$$\frac{\delta_{i-1}^-}{s_{i-1}} |f(a_{i-1})| + \frac{\delta_i^+}{s_{i+1}} |f(a_i)| \leq \max\left\{\frac{\delta_{i-1}^-}{s_{i-1}}, \frac{\delta_i^+}{s_{i+1}}\right\}\left(\overline{\bigvee}_{I_i} f + \frac{2}{m(I_i)}\int_{I_i}|f|\right).$$

Therefore,

$$\begin{aligned}\sum_{j=1}^{N} |P_\tau f(x_j) - P_\tau f(x_{j-1})| \quad \leq \quad & \frac{M}{s^2}\sum_{i=1}^{q}\int_{I_i}|f| + \sum_{i=1}^{q}\epsilon_i(N) + \sum_{i=1}^{q}\frac{1}{s_i}\overline{\bigvee}_{I_i} f + \\ & \max\left\{\frac{\delta_0^+}{s_1}, \frac{\delta_1^+}{s_2}\right\}\left(\overline{\bigvee}_{I_1} f + \frac{2}{m(I_1)}\int_{I_1}|f|\right) + \\ & \max\left\{\frac{\delta_{q-1}^-}{s_{q-1}}, \frac{\delta_q^-}{s_q}\right\}\left(\overline{\bigvee}_{I_q} f + \frac{2}{m(I_q)}\int_{I_q}|f|\right) + \\ & \sum_{i=2}^{q-1}\max\left\{\frac{\delta_{i-1}^-}{s_{i-1}}, \frac{\delta_i^+}{s_{i+1}}\right\}\left(\overline{\bigvee}_{I_i} f + \frac{2}{m(I_i)}\int_{I_i}|f|\right)\end{aligned}$$

Estimating I_i by $\min_i m(I_i)$ and combining appropriate terms together we get

$$\begin{aligned}\sum_{j=1}^{N} |P_\tau f(x_j) - P_\tau(x_{j-1})| \quad \leq \quad & \max_{1\leq i\leq q}\left\{\frac{1}{s_i} + \eta_i\right\}\bigvee_I f + \left[\frac{M}{s^2} + \frac{2\max_{1\leq i\leq q}\eta_i}{\min_i m(I_i)}\right]\int_I |f| dm + \\ & \sum_{i=1}^{q}\epsilon_i(N).\end{aligned}$$

Finally, letting $N \to \infty$, we arrive at inequality (3.1). □

Corollary 3.1 ("Standard" Lasota-Yorke inequality)**.** *If $s = \min_{1\leq i\leq q} s_i > 2$, then we obtain the standard Lasota-Yorke inequality (see [10])*

$$\bigvee_I P_\tau f \leq 2s^{-1}\bigvee_I f + (K + 2\beta^{-1})\|f\|_{\mathcal{L}^1},$$

where, $K := M/s^2$ and $\beta := \min_{1\leq i\leq q} m(I_i)$.

Remark 3.1. The classical proof of the standard Lasota-Yorke inequality differs from that of Proposition 3.1 in grouping the terms in the estimates. It uses the inequality

$$\bigvee_I P_\tau f \leq \sum_{i=1}^{q}\left(\bigvee_{\tau(I_i)} g_i f_i + \left|g_i(\tau(a_{i-1}^+))f_i(\tau(a_{i-1}^+))\delta_{i-1}^+\right| + \left|g_i(\tau(a_i^-))f_i(\tau(a_i^-))\delta_i^-\right|\right),$$

and then proceeds similarly as above without mixing the terms from the neighbouring subintervals. This works if $s_i > 2$ or if both δ_{i-1}^+ and δ_i^- are 0.

The following theorems state conditions under which the coefficient of $\bigvee_I f$ in inequality (3.1) is less than 1. That is,

$$\max_{1\leq i\leq q}\left\{\frac{1}{s_i} + \eta_i\right\} \leq \alpha < 1, \tag{3.3}$$

for some $\alpha > 0$.

Theorem 3.2. *Suppose $\tau \in \mathcal{T}(I)$ satisfies the following condition.*

$$\frac{1}{s_i} + \frac{1}{s_{i+1}} \le \alpha < 1, \text{ for } i = 1, \dots, q-1.$$

Then (3.3) holds for τ, or for an extension $(\hat{\tau}, \hat{I})$ of (τ, I) that contains (τ, I) as an attractor.

Proof. Let us first assume that

$$\tau(0), \tau(1) \in \{0, 1\}. \tag{3.4}$$

For $i = 1$ (and similarly for $i = q$):

$$\frac{1}{s_1} + \eta_1 \le \frac{1}{s_1} + \max\left\{\frac{\delta_0^+}{s_1}, \frac{\delta_1^+}{s_2}\right\} \le \frac{1}{s_1} + \frac{1}{s_2} \le \alpha < 1,$$

For $i = 2, \dots, q-1$:

$$\frac{1}{s_i} + \eta_i \le \frac{1}{s_i} + \max\left\{\frac{\delta_{i-1}^-}{s_{i-1}}, \frac{\delta_i^+}{s_{i+1}}\right\} \le \max\left\{\frac{1}{s_i} + \frac{1}{s_{i-1}}, \frac{1}{s_i} + \frac{1}{s_{i+1}}\right\} \le \alpha < 1,$$

If condition (3.4) does not hold, we extend the map τ to a map $\hat{\tau}$ defined on a larger interval $\hat{I}$ for which condition (3.4) is satisfied and the original system (τ, I) is an attractor. The idea of the proof is presented in Figure 1. □

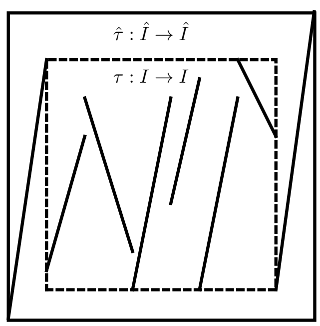


FIGURE 1. Extending the system $\tau\colon I \to I$ to $\tau\colon \hat{I} \to \hat{I}$ in such a way that (τ, I) is the attractor of $(\hat{\tau}, \hat{I})$.

The following theorem is a generalization of Theorem 3.2.

Theorem 3.3. *Let $s_i^* := \frac{s_i}{s_i - 1}$. Suppose $\tau \in \mathcal{T}(I)$ satisfies the following conditions.*

(1) For the first branch
  (a) if $\tau(a_0) \notin \{0, 1\}$ then $s_1 > 2$;
  (b) if $\tau(a_1^-) \notin \{0, 1\}$, then $s_2 > s_1^*$.
(2) For the last branch
  (a) if $\tau(a_{q-1}^+) \notin \{0, 1\}$ then $s_{q-1} > s_q^*$;
  (b) if $\tau(a_q^-) \notin \{0, 1\}$, then $s_q > 2$.
(3) For all $i = 2, \dots, q-1$,
  (a) if $\tau(a_{i-1}^+) \notin \{0, 1\}$ then $s_{i-1} > s_i^*$;

(b) if $\tau(a_i^-) \notin \{0,1\}$, then $s_{i+1} > s_i^*$.

Then (3.3) holds.

Proof. Let us sketch a proof of (3.3). First, note that we can find $0 < \alpha < 1$ such that for all $i = 1, \dots, q-1$,

(i) if $s_{i+1} > s_i^*$, then $s_{i+1} \geq \frac{s_i}{\alpha s_i - 1} > s_i^*$;
(ii) if $s_1 > 2$, then $s_1 \geq \frac{2}{\alpha} > 2$; and,
(iii) if $s_q > 2$, then $s_q \geq \frac{2}{\alpha} > 2$.

Now suppose $i = 1$. Conditions (i), (ii) above, and condition (1a) of the theorem imply

$$\frac{1}{s_1} + \eta_1 \leq \max\left\{\frac{2}{s_1}, \frac{1}{s_1} + \frac{1}{s_2}\right\} \leq \alpha < 1.$$

A similar argument applies when $i = 2, \dots, q$ showing

$$\frac{1}{s_i} + \eta_i \leq \alpha < 1,$$

hence proving (3.3). ☐

4. Existence and stability of acim's

Our existence and stability results are the applications of known results and methods to a wider space of maps. By a density we mean a function $f \in \mathcal{L}^1$ such that $f \geq 0$ and $\int f dm = 1$.

Theorem 4.1 (quasicompactness and existence of acim's)**.** *If a map $\tau \in \mathcal{T}(I)$ satisfies inequality (3.1) with the coefficient*

$$\max_{1\leq i\leq q}\left\{\frac{1}{s_i} + \eta_i\right\} \leq \alpha < 1,$$

for some $\alpha > 0$, then for any $f \in BV(I)$ and $n \in \mathbb{N}$,

$$\|P_\tau^n f\|_{BV} \leq \alpha^n \|f\|_{BV} + \left(1 + \frac{K + 2\beta^{-1}}{1-\alpha}\right)\|f\|_{\mathcal{L}^1},$$

where $K := M/s^2$ and $\beta := \min_{1\leq i\leq q} m(I_i)$. Furthermore, τ admits an acim with a density of bounded variation and $P_\tau : BV(I) \to BV(I)$ is quasicompact.

Proof. Using the norm $\|\cdot\|_{BV} := \bigvee_I(\cdot) + \|\cdot\|_{\mathcal{L}^1}$ of $BV(I)$ and Proposition 3.1, it follows that for all $n \in \mathbb{N}$,

$$\|P_\tau^n f\|_{BV} \leq \alpha^n \|f\|_{BV} + \left(1 + \frac{K + 2\beta^{-1}}{1-\alpha}\right)\|f\|_{\mathcal{L}^1}.$$

Since $\{f \in BV : \|f\|_{BV} \leq 1\}$ is relatively compact in the $\|\cdot\|_{\mathcal{L}^1}$ norm, it follows by standard arguments that P_τ has a fixed point in $BV(I)$, the essential spectral radius of P_τ (as defined below) on $BV(I)$ is $\leq \alpha < 1$ and $P_\tau : BV(I) \to BV(I)$ is quasicompact. ☐

Definition 4.2. (essential spectral radius) Consider a bounded linear operator P. We denote its spectrum by $\sigma(P)$ and its spectral radius by r_{spec}. The set of all eigenvalues in $\sigma(P)$ that are isolated and of finite multiplicity will be called the discrete spectrum of P, denoted by $\sigma_{disc}(P)$. The complement of $\sigma_{disc}(P)$ in $\sigma(P)$ will be called the essential spectrum of P, denoted by $\sigma_{ess}(P)$. The essential spectral radius r_{ess} of P is defined as the smallest upper bound for all elements of $\sigma_{ess}(P)$.

Definition 4.3. (eigenvalue gap, spectral gap) The eigenvalue gap of P is defined as $r_{spec} - \lambda_2$, where $\lambda_2 := \sup\{\lambda \in \sigma(P) : |\lambda| < r_{spec}(P)\}$. The spectral gap of P is defined as $r_{spec} - r_{ess}$.

For any $\tau \in \mathcal{T}$, it is well-known that the spectral radius of P_τ on the space $BV(I)$ is equal to 1. That is, the eigenvalue gap of P_τ equals $1 - \lambda_2$ while its spectral gap equals $1 - r_{ess}$.

Now we turn to the problem of stability. We shall use the Skorokhod metric as a measure of closeness for maps.

Definition 4.4. (Skorokhod metric) The Skorokhod distance $d_S(\tau_\epsilon, \tau_0)$ between two maps is the infimum of all positive r such that there exists a subset $A_r \subseteq I$ with $m(A_r) > 1 - r$ and a diffeomorphism $\sigma : I \to I$ such that

$$\tau_{\epsilon|_{A_r}} = \tau_0 \circ \sigma_{|_{A_r}}, \ |\sigma(x) - x| < r, \text{ and } \left|\frac{1}{\sigma'(x)} - 1\right| < r,$$

for all $x \in A_r$.

The following stability theorem is a direct consequence of Keller-Liverani stability results (see e.g. Corollary 1, 2 and Remark 4 of [7]) and Proposition 3.1.

Theorem 4.5 (stability)**.** *Consider the one-parameter family of maps $\{\tau_\epsilon\}_{\epsilon \geq 0}$, where $\{\tau_\epsilon\}_{\epsilon \geq 0} \in \mathcal{T}(I)$ uniformly. Suppose there exists $0 < \alpha < 1$ such that*

$$\max_{1 \leq i \leq q} \left\{ \frac{1}{s_i} + \eta_i \right\} \leq \alpha < 1.$$

Let f_ϵ be a τ_ϵ-invariant density whose existence is guaranteed by Theorem 4.1. If $d_S(\tau_\epsilon, \tau_0) \to 0$ as $\epsilon \to 0$, then the following statements hold.

1. *The family $\{f_\epsilon\}_{\epsilon > 0}$ is relatively compact in $\mathcal{L}^1$ and any of its limit functions is a τ_0-invariant density.*
2. *If τ_0 is ergodic, then τ_ϵ is ergodic for small ϵ and $f_\epsilon \to f_0$ in $\mathcal{L}^1$ as $\epsilon \to 0$ (i.e. τ_0 is acim-stable).*
3. *If τ_0 is weakly mixing, then the eigenvalue gaps of $\{P_{\tau_\epsilon}\}_\epsilon$, for ϵ small enough, are uniformly bounded, i.e. $0 < \gamma < 1 - |\lambda_2^\epsilon|$. As a consequence, there exists a constant $C > 0$ such that for all ϵ small enough and all densities $f \in BV$*

$$\left\| P^n_{\tau_\epsilon} f - f_\epsilon \right\|_{\mathcal{L}^1} \leq C(1 - \gamma)^n \|f\|_{BV}. \tag{4.1}$$

Remark 4.6. It also follows by Theorem 4.1 that there is a uniform spectral gap for the family $\{P_{\tau_\epsilon}\}_{\epsilon \geq 0}$ bounded below by $1 - \alpha$.

The stronger L-Y inequality (3.1) allows us to apply the results about stochastic perturbations like those discussed in [4] and [6] to a wider class of maps satisfying conditions of Theorems 3.2 or 3.3. In particular, inequality (3.1) extends the validity of Ulam's approximation method to such maps. Similarly, the results of [12] can be extended to this class of maps.

5. Examples

Below we give examples of situations ensuring that the assumptions of Theorem 4.5 are satisfied.

Example 5.1. Assume that $\tau_0 \in \mathcal{T}(I)$ and satisfies condition (3.3). Assume that $\{\tau_\epsilon\}_{\epsilon>0}$ is defined on the same partition $\mathcal{P} = \{I_1, I_2, \ldots, I_q\}$ as τ_0, and $\tau_\epsilon \to \tau_0$ as $\epsilon \to 0$ in $\mathcal{C}^1(\text{int}(I_i))$ for all $i = 1, 2, \ldots, q$. Then, $d_S(\tau_\epsilon, \tau_0) \to 0$ as $\epsilon \to 0$, $\{\tau_\epsilon\} \subset \mathcal{T}(I)$ uniformly for all $\epsilon \geq 0$, and the conclusions of Theorem 4.5 hold.

Example 5.2. Assume $\tau_0 \in \mathcal{T}(I)$ and satisfies condition (3.3). Assume that τ_ϵ is piecewise expanding on the partition $\mathcal{P}_\epsilon = \{I_1^{(\epsilon)}, I_2^{(\epsilon)}, \ldots, I_q^{(\epsilon)}\}$, $I_i^{(\epsilon)} = (a_{i-1}^{(\epsilon)}, a_i^{(\epsilon)})$, such that $a_i^{(\epsilon)} \to a_i^{(0)}$ as $\epsilon \to 0$ (in particular, τ_ϵ has the same number of monotonic branches as τ_0). Additionally, assume that there exists $\epsilon_1 > 0$ such that for every $0 < \epsilon_0 < \epsilon_1$, $\tau_\epsilon \to \tau_0$ in $\mathcal{C}^1$ on the set

$$\bigcup_{i=1,2,\ldots,q} \left[\max\left\{a_{i-1}^{(0)}, a_{i-1}^{(\epsilon_0)}\right\}, \min\left\{a_i^{(0)}, a_i^{(\epsilon_0)}\right\}\right],$$

and that $\{\tau_\epsilon\} \subset \mathcal{T}(I)$ uniformly for all $\epsilon \geq 0$. Then, $d_S(\tau_\epsilon, \tau_0) \to 0$ as $\epsilon \to 0$ and the conclusions of Theorem 4.5 hold.

Example 5.3 (asymmetric W-map)**.** Let W_0 be the asymmetric W-map whose graph is shown in Figure 2(a). It is straightforward to check that W_0 satisfies the slope conditions of Theorem 3.2. Therefore, it is acim-stable with respect to perturbations described in examples 5.1 and 5.2.

Remark 5.4. W-maps were first constructed by Keller [6] and shown to be acim-unstable under perturbations that force the existence of invariant intervals (e.g. perturbations that only move the fixed turning point downward). It was shown in [2] that Markov W-maps could be acim-unstable even with respect to perturbations that do not produce invariant interval (e.g. perturbations that move the fixed turning point upward). Later in [11] the acim-instability of these maps was proven without assuming that they are Markov. In both of the papers the limiting W-map had slope $= 2$ on both sides of the fixed turning point. Recently, in [13] and using the same techniques as [11], it was shown that these maps are acim-unstable (under specific perturbations) if $1/s_2 + 1/s_3 \geq 1$, and acim-stable if $1/s_2 + 1/s_3 < 1$. Here s_2 and s_3 represent the slopes on the left and right side of the fixed turning point, respectively.

Example 5.5. Consider the piecewise linear map τ with five branches whose graph is shown in Figure 2(b). The partition points are $\left\{0, \frac{1}{20}, \frac{1}{10}, \frac{1}{4}, \frac{1}{3}, 1\right\}$ and the slopes $\left\{16, 16, \frac{5}{3}, 3, \frac{3}{2}\right\}$, correspondingly. It is easily checked that τ satisfies the hypothesis of Theorem 3.3. In fact,

$$\max_{1\leq i\leq 5}\left\{\frac{1}{s_i} + \eta_i\right\} = \max\left\{\frac{1}{16} + \frac{1}{16}, \frac{1}{16} + \frac{3}{5}, \frac{3}{5} + \frac{1}{3}, \frac{1}{3} + \frac{3}{5}, \frac{2}{3} + 0\right\} = \frac{14}{15} < 1.$$

Therefore, τ is acim-stable with respect to perturbations described in examples 5.1 and 5.2.

The results of this paper allow to answer a question posed in [2].

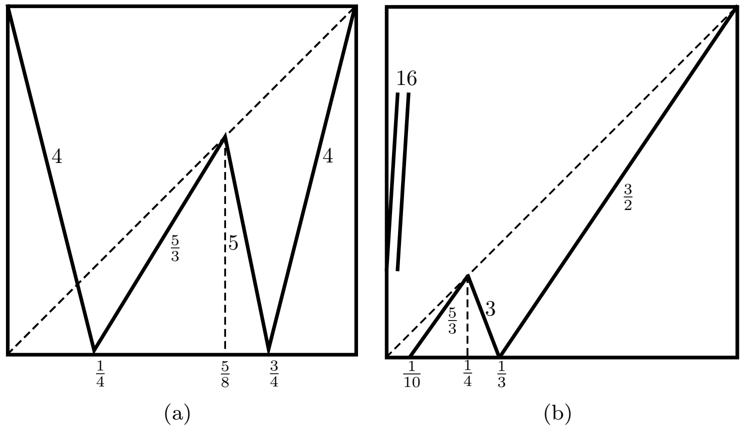


FIGURE 2. Figure (a) shows the graph of the asymmetric W-map of Example 5.3. Figure (b) shows the graph of the map τ of Example 5.5.

Example 5.6. Consider the family of unimodal maps $\{\tau_t\}_{0\le t<1/2}$ defined by

$$\tau_t(x) = \begin{cases} \frac{1}{2} - t + (1+2t)x & 0 \le x < \frac{1}{2}, \\ 2 - 2x & \frac{1}{2} \le x \le 1. \end{cases} \tag{5.1}$$

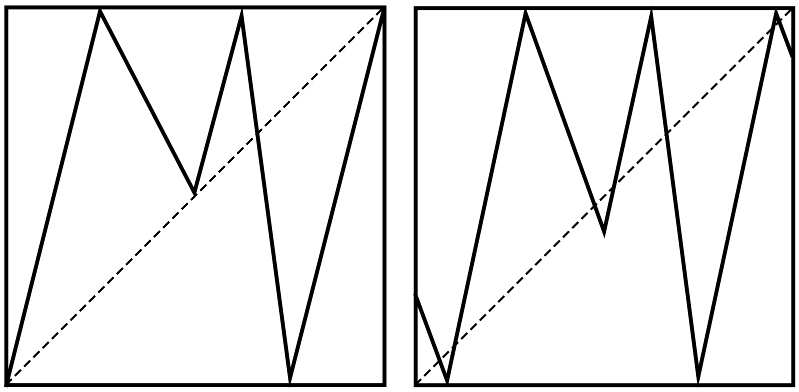
τ_0 is exact with invariant density $f_0 = \frac{2}{3}\chi_{[0,1/2]} + \frac{4}{3}\chi_{[1/2,1]}$. The question is whether τ_0 is acim-stable in the family $\{\tau_t\}_{t\ge 0}$? Note that τ_0 has a turning point at 1/2, which is periodic with period 3. Previously known methods did not give an answer to this question.

We will consider the family of third iterates $\{\tau_t^3\}_{t>0}$. The slopes of τ_t^3 are $s_1 = s_3 = s_7 = 2 + 8t + 8t^2$, $s_2 = s_4 = s_6 = 4 + 8t$, and $s_5 = 8$. The map τ_0^3 is shown in Figure 3(a) and a typical τ_t^3 is shown in Figure 3(b). Since τ_0 is exact, τ_0^3 is also exact with the same acim; moreover, the acim-stability of τ_0^3 implies the same for τ_0. Because of arbitrarily short intervals in the partitions $\mathcal{P}_t$ our results cannot be applied directly to the family $\{\tau_t^3\}$.

Let $g_1^{(t)}$ and $g_7^{(t)}$ be linear functions which coincide with the first and last branches of τ_t^3, respectively. For each t we find points $a_0^{(t)}$ and $a_8^{(t)}$ such that $g_1^{(t)}(a_0^{(t)}) = a_8^{(t)}$ and $g_7^{(t)}(a_8^{(t)}) = a_0^{(t)}$. We extend maps τ_t^3 to $[a_0^{(t)}, a_8^{(t)}]$ using the functions $g_1^{(t)}$ and $g_7^{(t)}$. Let us call the new maps $\hat{\tau}_t^3$ although they may not be third iterates of some other maps. The new maps are shown in Figure 4(b) for $t > 0$ and in Figure 4(a) for the limiting case $t = 0$.

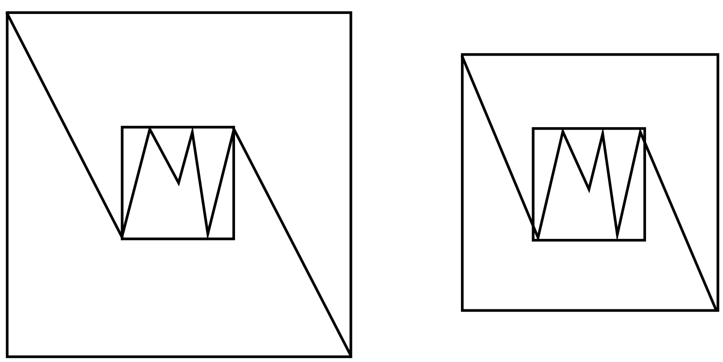
The extended family satisfies assumptions of Theorem 3.2 and Example 5.2, so we have acim-stability as described in Theorem 4.5. For all maps $\hat{\tau}_t^3$ the interval $[0, 1]$ is the attractor supporting the unique acim's. Thus, we obtain acim-stability of τ_0^3 and consequently the acim-stability of τ_0.

Acknowledgments. We thank the anonymous referee for patiently correcting our proofs and for suggesting improvements to our paper. The first author would also like to thank Dr. Oliver Butterley for his valuable remarks on an earlier version of the paper and Professor Carlangelo Liverani for informing us of the recent preprint [9]. The first author was supported by the INdAM-COFUND Marie Curie Fellowship during the final stages of the preparations of this article.



(a) (b)

FIGURE 3. Figure (a) shows the graph of τ_0^3 of Example 5.6. Figure (b) shows the graph of τ_t^3, for $t = 0.1$.



(a) (b)

FIGURE 4. Figure (a) shows the graph of $\hat{\tau}_0^3$ of Example 5.6. Figure (b) shows the graph of $\hat{\tau}_t^3$, for $t = 0.05$.

REFERENCES

1. A. Boyarsky and P. Góra, *Laws of Chaos. Invariant Measures and Dynamical Systems in One Dimension*, Probability and its Applications, Birkhäuser, Boston, MA, 1997.
2. P. Eslami and M. Misiurewicz, Singular limits of absolutely continuous invariant measures for families of transitive map, *J. Difference Equ. Appl.* **18** (2012) 739–750.
3. E. Giusti, *Minimal Surfaces and Functions of Bounded Variations*, Brikhaüser, Boston, 1984.
4. P. Góra, On small stochastic perturbations of mappings of the unit interval. *Colloq. Math.*, **49** (1984) 73–85.

5. P. Góra and B. Schmitt, Un exemple de transformation dilatante et $\mathcal{C}^1$ par morceaux de l'intervalle, sans probabilité absolument continue invariante, *Ergod. Th. and Dynami. Sys.*, **9** (1989) 101–113.
6. G. Keller, Stochastic stability in some chaotic dynamical systems, *Monatshefte für Mathematik*, **94** (4) (1982) 313–333.
7. G. Keller and C. Liverani, Stability of the spectrum for transfer operators, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4), **28** (1)(1999), 141–152.
8. Z. S. Kowalski, Continuity of invariant measures for piecewise monotonic transformations. *Bull. Acad. Polon. Sci. Sr. Sci. Math. Astronom. Phys.*, **23** (1975) 519–524.
9. C. Liverani, Multidimensional expanding maps with singularities: a pedestrian approach. To appear in *Ergodic Theory and Dynam. Systems*, **33** (2013) 168–182.
10. A. Lasota and J. A. Yorke, On the existence of invariant measures for piecewise monotonic transformations, *Trans. Amer. Math. Soc.*, 186 (1973) 481–488.
11. Zh. Li, P. Gora, A. Boyarsky, H. Proppe and P. Eslami, A Family of Piecewise Expanding Maps having Singular Measure as a limit of ACIM's, *Ergodic Theory and Dynamical Systems* **33** (2013) 158–167.
12. C. Tokman, B. Hunt and P. Wright, Approximating invariant densities of metastable systems, *Ergodic Theory and Dynam. Systems*, **31** 1345–1365.
13. Zhenyang Li, W-like maps with various instabilities of acim's, *Internat. J. Bifur. Chaos Appl. Sci. Engrg.* **23** (2013.)

Peyman Eslami, Dipartimento di Matematica, II Università di Roma (Tor Vergata), Via della Ricerca Scientifica, 00133 Roma, Italy.

E-mail address: eslami@mat.uniroma2.it

Department of Mathematics & Statistics, Concordia University, 1455 de Maisonneuve Blvd. West, Montreal, Quebec H3G 1M8, Canada

E-mail address: pgora@mathstat.concordia.ca