METASTABLE SYSTEMS AS RANDOM MAPS

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Abstract. Metastable dynamical systems were recently studied [9] in the framework of one-dimensional piecewise expanding maps on two disjoint invariant sets, each possessing its own ergodic absolutely continuous invariant measure (acim). Under small deterministic perturbations, holes between the two disjoint systems are created, and the two ergodic systems merge into one. The long term dynamics of the newly formed metastable system is defined by the unique acim on the combined ergodic sets. The main result of [9] proves that this combined acim can be approximated by a convex combination of the disjoint acims with weights depending on the ratio of the respective measures of the holes. In this note we present an entirely different approach to metastable systems. We consider two piecewise expanding maps: one is the original map, τ_1, defined on two disjoint invariant sets of $\mathbb{R}^N$ and the other is a deterministically perturbed version of τ_1, τ_2, which allows passage between the two disjoint invariant sets of τ_1. We model this system by a position dependent random map based on τ_1 and τ_2, to which we associate position dependent probabilities that reflect the switching between the maps. A typical orbit spends a long time in one of the ergodic sets but eventually switches to the other. Such behavior can be attributed to physical holes as between adjoining billiard tables or more abstract situations where balls can "leap" from one table to the other. Using results for random maps a result similar to the one dimensional main result of [9] is proved in N dimensions. We also consider holes in more than two invariant sets. A number of examples are presented.

1. Introduction

One-dimensional metastable systems were recently studied [9] in the framework of piecewise expanding maps on two disjoint ergodic sets. Under small deterministic perturbations, the asymptotic dynamics of the merged metastable system is captured by the absolutely continuous invariant measure (acim) on the combined ergodic sets. The main result of [9] shows that this combined acim can be approximated by a convex combination of the two disjoint acims with weights depending on the respective measures of the holes. The method of [9] invokes the usual BV techniques that apply naturally in a setting where the slopes of the original map are > 2. For maps with slopes only > 1 in magnitude, the BV technique encounters difficulties as the partitions needed for the approximating family of maps have elements that go to zero in measure and hence render the standard BV inequalities ineffective in establishing precompactness of the family of probability density functions associated with the family of approximating maps. To handle this problem, the authors of [9] introduce some additional conditions on the maps they consider.

In this note we take a different approach to modeling metastable behavior. We consider two piecewise expanding maps: one is the original map, τ_1, defined on two disjoint invariant sets of $\mathbb{R}^N$ and the other, τ_2, is a deterministically perturbed version of τ_1, τ_2, which allows passage between the two disjoint invariant sets of τ_1 via holes. We model such a system by means of a random map based on τ_1 and τ_2, to which we associate position dependent

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probabilities that reflect the switching between the maps. A typical orbit spends a random amount of time governed by the dynamics of either τ_1 or τ_2, then switches to the other map. Suppose p_1, the probability of using map τ_1, is close to 1, then, with very high probability, the orbit spends a lot of time under the influence of τ_1, that is, it stays in either one or the other of the two disjoint sets invariant under τ_1. Since $p_1 < 1$, there is a small but positive probability of switching from τ_1 to τ_2. When this happens, the dynamics comes under the control of τ_2, which allows movement between the disjoint invariant sets. Unlike the model in [9] where the hole sizes shrink to 0, the hole sizes in our random map model stay fixed. (Their measures in a skew product interpretation of random maps converge to 0, so one could argue that both models are in a way similar.) What changes are the probabilities of switching from one map to the other. As p_1 approaches 1, the orbits are almost completely defined by τ_1 and therefore remain in one or the other of the two disjoint invariant sets for a very long time. This behavior is the hallmark of metastable dynamics. Our main result establishes a result similar in spirit to that in [9]: we prove that, as the probability of using τ_1 converges to 1, the dynamics are captured by an acim that is a convex combination of the acims on the two disjoint invariant sets and we can calculate the weights of the respective acims from a formula analogous to the one derived in [9].

In the billiards problem metastable behavior is attributed to small physical holes in the boundary between the tables. From the perspective of random maps, the holes can be large; it is the probabilities of switching that controls the metastable behavior. This allows for the consideration of situations where there are no actual physical holes, but where balls can "leap" from one table to the other.

In Sections 2 and 3 we recall the definition of a position dependent random map and collect some existence and continuity results in 1 and N dimensions. In Section 4 we present the main result for holes between two invariant sets: there exists a unique acim which is a convex combination of the acims on the two ergodic sets and the weights in the combination can be calculated from a formula similar to the one in [9]. In Section 5 we present the main result for a case with more than two invariant subsystems. Deterministic model of such situation is discussed in [4]. Section 6 contains examples.

2. Position Dependent Random Maps and Their Properties

Let $(I, \mathfrak{B}, \lambda)$ be a measure space, where λ is an underlying measure. Let $\tau_k : I \to I$, $k = 1, ..., K$ be piecewise one-to-one, differentiable, non-singular transformations on a common partition $\mathcal{P}$ of I : $\mathcal{P} = \{I_1, ..., I_q\}$ and $\tau_{k,i} = \tau_k \mid_{I_i}$, $i = 1, ..., q$, $k = 1, ..., K$ ($\mathcal{P}$ can be found by considering finer partitions). We define the transition function for the random map $T = \{\tau_1, ...\tau_K; p_1(x), ...p_K(x)\}$ as follows:

$$\mathbb{P}(x, A) = \sum_{k=1}^{K} p_k(x)\chi_A(\tau_k(x)), \tag{2.1}$$

where A is any measurable set and $\{p_k(x)\}_{k=1}^{K}$ is a set of position dependent measurable probabilities, i.e., $\sum_{k=1}^{K} p_k(x) = 1$, $p_k(x) \geq 0$, for any $x \in I$ and χ_A denotes the characteristic function of the set A. We define $T(x) = \tau_k(x)$ with probability $p_k(x)$ and $T^N(x) = \tau_{k_N} \circ \tau_{k_{N-1}} \circ ... \circ \tau_{k_1}(x)$ with probability $p_{k_N}(\tau_{k_{N-1}} \circ ... \circ \tau_{k_1}(x)) \cdot p_{k_{N-1}}(\tau_{k_{N-2}} \circ ... \circ \tau_{k_1}(x)) \cdots p_{k_1}(x)$.

The transition function $\mathbb{P}$ induces an operator $\mathbb{P}_*$ on measures on $(I, \mathfrak{B})$ defined by

$$
\begin{aligned}
\mathbb{P}_*\mu(A) &= \int_I \mathbb{P}(x, A) d\mu(x) = \sum_{k=1}^{K} \int_I p_k(x)\chi_A(\tau_k(x)) d\mu(x) \\
&= \sum_{k=1}^{K} \int_{\tau_k^{-1}(A)} p_k(x) d\mu(x) = \sum_{k=1}^{K}\sum_{i=1}^{q} \int_{\tau_{k,i}^{-1}(A)} p_k(x) d\mu(x)
\end{aligned} \tag{2.2}
$$

We say that the measure μ is T-invariant iff $\mathbb{P}_*\mu = \mu$, i.e.,

$$
\mu(A) = \sum_{k=1}^{K} \int_{\tau_k^{-1}(A)} p_k(x) d\mu(x), \qquad A \in \mathfrak{B}. \tag{2.3}
$$

If μ has density f with respect to λ, the $\mathbb{P}_*\mu$ has also a density which we denote by $P_T f$. By change of variables, we obtain

$$
\begin{aligned}
\int_A P_T f(x) d\lambda(x) &= \sum_{k=1}^{K}\sum_{i=1}^{q} \int_{\tau_{k,i}^{-1}(A)} p_k(x) f(x) d\lambda(x) \\
&= \sum_{k=1}^{K}\sum_{i=1}^{q} \int_A p_k(\tau_{k,i}^{-1}x) f(\tau_{k,i}^{-1}x) \frac{1}{J_{k,i}(\tau_{k,i}^{-1})} d\lambda(x) \ ,
\end{aligned} \tag{2.4}
$$

where $J_{k,i}$ is the Jacobian of $\tau_{k,i}$ with respect to λ, $J(\tau) = \frac{d\tau_*(\lambda)}{d\lambda}$. Since this holds for any measurable set A we obtain an a.e. equality:

$$
(P_T f)(x) = \sum_{k=1}^{K}\sum_{i=1}^{q} p_k(\tau_{k,i}^{-1}x) f(\tau_{k,i}^{-1}x) \frac{1}{J_{k,i}(\tau_{k,i}^{-1})} \chi_{\tau_k(I_i)}(x) \tag{2.5}
$$

or

$$
(P_T f)(x) = \sum_{k=1}^{K} P_{\tau_k}(p_k f)(x) \tag{2.6}
$$

where P_{τ_k} is the Perron-Frobenius operator corresponding to the transformation τ_k (see [2] for details). We call P_T the Perron-Frobenius operator of the random map T.

3. Continuity theorems

3.1. **Existence and continuity theorem in one dimension.** Let $(I, \mathfrak{B}, \lambda)$ be a measure space, where λ is normalized Lebesgue measure on $I = [a, b]$. Let $\tau_k : I \to I$, $k = 1, ..., K$ be piecewise one-to-one and C^2, non-singular transformations on a partition $\mathcal{P}$ of I : $\mathcal{P} = \{I_1, ..., I_q\}$ and $\tau_{k,i} = \tau_k \mid_{I_i}$, $i = 1, ..., q$, $k = 1, ..., K$. Let $\{p_k(x)\}_{k=1}^K$ be a set of position dependent measurable probabilities, i.e., $\sum_{k=1}^K p_k(x) = 1$, $p_k(x) \geq 0$, for any $x \in I$. Assume in addition that p_k is piecewise differentiable on $\mathcal{P}$.

Denote by $V(\cdot)$ the standard one-dimensional variation of a function, and by $BV(I)$ the space of functions of bounded variations on I equipped with the norm $\|\cdot\|_{BV} = V(\cdot) + \|\cdot\|_1$.

Let $g_k(x) = \frac{p_k(x)}{|\tau_k'(x)|}$, $k = 1, \ldots, K$. We assume the following conditions:

Condition (A): $\sum_{k=1}^K g_k(x) < \alpha < 1$, $x \in I$, and
Condition (B): $g_k \in BV(I)$, $k = 1, \ldots, K$.

Let $T = \{\tau_1, \dots, \tau_K; p_1, \dots, p_K\}$ be a random map with position dependent probabilities satisfying conditions (A) and (B). We define $\mathcal{P}^N$ as a maximal common monotonicity partition for all maps defining T^N. For $w = (k_1, \dots, k_{N-1}, k_N) \in \{1, ..., K\}^N$ we define

$$g_w = \frac{p_{k_N}(\tau_{k_{N-1}} \circ ... \circ \tau_{k_1}(x)) \cdot p_{k_{N-1}}(\tau_{k_{N-2}} \circ ... \circ \tau_{k_1}(x)) \cdots p_{k_1}(x)}{|(\tau_{k_N} \circ \tau_{k_{N-1}} \circ ... \circ \tau_{k_1})'(x)|}.$$

The following results are proved in [1]:

Lemma 3.1. *Let T satisfy conditions (A) and (B). Then for any $f \in BV(I)$ and $M \in \mathbb{N}$,*

$$\|P_T^M f\|_{BV} \leq A_M \|f\|_{BV} + B_M \|f\|_1, \tag{3.1}$$

where $A_M = 3\alpha^M + W_M$, $B_M = \beta_M(2\alpha^M + W_M)$, $\beta_M = \max_{J \in \mathcal{P}^M} (\lambda(J))^{-1}$, $W_M \equiv \max_{J \in \mathcal{P}^M} \sum_{w \in \{1,...,K\}^M} V_J g_w$.

Theorem 3.2. *Let T be a random map which satisfies conditions (A) and (B). Then T preserves a measure which is absolutely continuous with respect to Lebesgue measure. The operator P_T is quasi-compact on $BV(I)$, see* [2].

We now present the continuity theorem in one dimension. A similar theorem was proved in proposition 2 of [6] under stronger conditions. Our aim is to show that it holds under the weaker conditions (A) and (B).

Theorem 3.3 (Continuity Theorem 1-dim). *Let $T = \{\tau_1, \dots, \tau_K; p_1, \dots, p_K\}$ be a random map with position dependent probabilities satisfying conditions (A) and (B). Let $\{p_1^{(n)}, \dots, p_K^{(n)}\}_{n=1}^{\infty}$ be a sequence of sets of probabilities such that $p_k^{(n)} \to p_k$ as $n \to +\infty$, $k = 1, \dots, K$, in the piecewise C^1 topology on the fixed partition $\mathcal{P}$. Let $T^{(n)} = \{\tau_1, \dots, \tau_K; p_1^{(n)}, \dots, p_K^{(n)}\}$, $n = 1, 2, \dots$ be a sequence of random maps. For n large, $T^{(n)}$ has an invariant density $f^{(n)}$ and the sequence $\{f^{(n)}\}_{n=1}^{\infty}$ is precompact in L^1. Moreover, any limit point f^* of this sequence is a fixed point of P_T.*

Proof. We will prove the theorem in three steps. In the first step we show that an inequality similar to inequality (3.1) of lemma 3.1 holds uniformly for all $T^{(n)}$ with n large enough. In order to achieve this, we need to show that for large enough n conditions (A) and (B) are satisfied uniformly.

Suppose $\alpha < \gamma < 1$, where $\sum_{k=1}^K g_k(x) < \alpha < 1$. First, choose ϵ such that $\sum_{k=1}^K \frac{\epsilon}{|\tau_k'(x)|} < \gamma - \alpha$. Then choose N_1 such that for $n > N_1$ and $1 \leq k \leq K$, $p_k - \epsilon \leq p_k^{(n)} \leq p_k + \epsilon$. Then

$$\sum_{k=1}^K \frac{p_k^{(n)}(x)}{|\tau_k'(x)|} \leq \sum_{k=1}^K \frac{p_k + \epsilon}{|\tau_k'(x)|} = \sum_{k=1}^K \frac{p_k(x)}{|\tau_k'(x)|} + \sum_{k=1}^K \frac{\epsilon}{|\tau_k'(x)|} \leq \alpha + (\gamma - \alpha) = \gamma < 1.$$

Therefore, condition (A) holds uniformly for all $n > N_1$, with α replaced by γ. Regarding condition (B), note that

$$|V_J g^{(n)} - V_J g| \leq \int_J |(g^{(n)})' - g'| d\lambda \to 0 \text{ as } n \to \infty.$$

It follows that there exists a constant C_1 and an integer N_2 such that for all $n > N_2$, $V_J g^{(n)} < C_1$ for any interval $J \subset I$.

Now consider $W_1^{(n)} = \max_{J \in \mathcal{P}} \sum_{k=1}^K V_J g_k^{(n)}$. From the above statement it follows that $W_1^{(n)}$ is also uniformly bounded for n sufficiently large. That is, there exists C_2 and integer N_3

such that for all $n > N_3$, $W_1^{(n)} < C_2$. Let $N_4 = \max\{N_1, N_2, N_3\}$ and $C = \max\{C_1, C_2\}$. It is shown in [1] that $W_M^{(n)} \le M\alpha^{M-1}W_1^{(n)}$, hence for $n > N_4$, $W_M^{(n)} < M\gamma^{M-1}C$. Therefore, for $n > N_4$, inequality (3.1) holds uniformly with α replaced by γ.

In the next step we show that the sequence of invariant densities $\{f^{(n)}\}$ is uniformly bounded in $BV(I)$. Without loss of generality consider $\{f^{(n)}\}_{n=N_4+1}^{\infty}$ instead of $\{f^{(n)}\}$. Moreover since inequality (3.1) is now satisfied uniformly for all n, we drop the superscript of (n) and write A_M, B_M for $A_M^{(n)}$, $B_M^{(n)}$ respectively. Also assume M is large enough so that $A_M = 3\gamma^M + W_M < 1$.

To summarize, we have shown that there exists M such that for any $f \in BV(I)$ and $n \in \mathbb{N}$:

$$\|P_{T^{(n)}}^M f\|_{BV} \le A_M\|f\|_{BV} + B_M\|f\|_1, \tag{3.2}$$

where $A_M = 3\gamma^M + W_M < 1$, $B_M = \beta_M(2\gamma^M + W_M)$, $\beta_M = \max_{J\in\mathcal{P}^M}(\lambda(J))^{-1}$, $W_M \equiv \max_{J\in\mathcal{P}^M}\sum_{w\in\{1,\dots,K\}^M} V_J g_w$.

Using inequality (3.2) repeatedly, one can show that each $f^{(n)}$ is a limit point of the sequence of averages $\left\{\frac{1}{m}\sum_{j=0}^{m-1} P_{T^{(n)}}^{Mj}1\right\}$ and

$$\|f^{(n)}\|_{BV} \le 1 + \frac{B_M}{1 - A_M}.$$

Therefore $\{f^{(n)}\}$ is a bounded set in $BV(I)$ and hence it has a limit point f^* in L^1.

In the final step show that f^* is invariant under P_T:

$$\begin{aligned}
\|P_T f^* - f^*\|_1 &\le \|P_T f^* - P_{T^{(n)}} f^*\|_1 + \|P_{T^{(n)}} f^* - P_{T^{(n)}} f^{(n)}\|_1 \\
&+ \|P_{T^{(n)}} f^{(n)} - f^{(n)}\|_1 + \|f^{(n)} - f^*\|_1 \\
&= \|\sum_{k=1}^{K} P_{\tau_k}(p_k f^*) - \sum_{k=1}^{K} P_{\tau_k}(p_k^{(n)} f^*)\|_1 \\
&+ \|\sum_{k=1}^{K} P_{\tau_k}(p_k^{(n)} f^*) - \sum_{k=1}^{K} P_{\tau_k}(p_k^{(n)} f^{(n)})\|_1 \\
&+ \|P_{T^{(n)}} f^{(n)} - f^{(n)}\|_1 + \|f^{(n)} - f^*\|_1 \\
&\le \sum_{k=1}^{K} \|f^*(p_k - p_k^{(n)})\|_1 + \sum_{k=1}^{K} \|(f^* - f^{(n)})p_k^{(n)}\|_1 \\
&+ \|P_{T^{(n)}} f^{(n)} - f^{(n)}\|_1 + \|f^{(n)} - f^*\|_1
\end{aligned}$$

The third summand is 0 by definition of $f^{(n)}$. The other three converge to 0 since $f^{(n)} \to f^*$ and $p_k^{(n)} \to p_k$ as $n \to \infty$ in L^1 and L^∞, respectively. □

3.2. **Existence and continuity theorem in higher dimensions.** We now prove the continuity theorem in $\mathbb{R}^N$. Let S be a bounded region in $\mathbb{R}^N$ and λ_N be Lebesgue measure on S. Let $\tau_k : S \to S$, $k = 1, \dots, K$ be piecewise one-to-one and C^2, non-singular transformations on a partition $\mathcal{P}$ of S : $\mathcal{P} = \{S_1, \dots, S_q\}$ and $\tau_{k,i} = \tau_k|_{S_i}$, $i = 1, \dots, q$, $k = 1, \dots, K$. Let each S_i be a bounded closed domain having a piecewise C^2 boundary of finite $(N-1)$-dimensional measure. We assume that the faces of ∂S_i meet at angles bounded uniformly away from 0.

We will also assume that the probabilities $p_k(x)$ are piecewise C^1 functions on the partition $\mathcal{P}$. Let $D\tau_{k,i}^{-1}(x)$ be the derivative matrix of $\tau_{k,i}^{-1}$ at x. We assume:
Condition (C):

$$\max_{1\le i\le q}\sum_{k=1}^{K} p_k(x)\|D\tau_{k,i}^{-1}(\tau_{k,i}(x))\| < \sigma < 1.$$

The main tool of this section is the multidimensional notion of variation defined using derivatives in the distributional sense (see [5]):

$$V(f)=\int_{\mathbb{R}^N}\|Df\| = \sup\{\int_{\mathbb{R}^N} f\mathrm{div}(g)d\lambda_N : g=(g_1,...,g_N)\in C_0^1(\mathbb{R}^N,\mathbb{R}^N)\},$$

where $f\in L_1(\mathbb{R}^N)$ has bounded support, Df denotes the gradient of f in the distributional sense, and $C_0^1(\mathbb{R}^N,\mathbb{R}^N)$ is the space of continuously differentiable functions from $\mathbb{R}^N$ into $\mathbb{R}^N$ having a compact support. We will use the following property of variation which is derived from [5], Remark 2.14: If $f=0$ outside a closed domain A whose boundary is Lipschitz continuous, $f_{|A}$ is continuous, $f_{|\mathrm{int}(A)}$ is C^1, then

$$V(f)=\int_{\mathrm{int}(A)}\|Df\|d\lambda_N+\int_{\partial A}|f|d\lambda_{N-1},$$

where λ_{N-1} is the $(N-1)$-dimensional measure on the boundary of A. In this section we shall consider the Banach space (see [5], Remark 1.12),

$$BV(S)=\{f\in L_1(S): V(f)<+\infty\},$$

with the norm $\|f\|_{BV}=V(f)+\|f\|_1$.

Theorems 3.4 and 3.5 below were established in [7]. We refer the reader to [7] for proofs as well as the precise definitions of the functions $a(\cdot)$ and $\delta(\cdot)$. We just remark here that for a random map $T=\{\tau_1,\dots,\tau_K;p_1,\dots,p_K\}$ the functions a and δ are independent of the probabilities $\{p_1,\dots,p_K\}$.

Theorem 3.4. *If T is a random map which satisfies condition (C), then*

$$V(P_Tf)\le\sigma(1+1/a)V(f)+(M+\frac{\sigma}{a\delta})\|f\|_1, \tag{3.3}$$

where $a=\min\{a(S_i): i=1,\dots,q\}>0$, $\delta=\min\{\delta(S_i),: i=1,\dots,q\}>0$, $M_{k,i}=\sup_{x\in S_i}(Dp_k(x)-\frac{DJ_{k,i}}{J_{k,i}}p_k(x))$ *and* $M=\sum_{k=1}^K\max_{1\le i\le q}M_{k,i}$.

Theorem 3.5. *Let T be a random map which satisfies condition (C). If $\sigma(1+1/a)<1$, then T preserves a measure which is absolutely continuous with respect to Lebesgue measure. The operator P_T is quasi-compact on $BV(S)$, see [2].*

Now we present the multi-dimensional version of theorem 3.3. The proof of this theorem is similar to the proof of the one-dimensional continuity theorem hence we will only sketch the proof here.

Theorem 3.6 (Continuity Theorem in $\mathbb{R}^N$)**.** *Let $T=\{\tau_1,\dots,\tau_K;p_1,\dots,p_K\}$ be a random map with position dependent probabilities, satisfying condition (C). Also assume that $\sigma(1+1/a)<1$, where a is as in* [1]. *Let $\{p_1^{(n)},\dots,p_K^{(n)}\}_{n=1}^\infty$ be a sequence of sets of probabilities such that $p_k^{(n)}\to p_k$ as $n\to+\infty$, $k=1,\dots,K$, in the piecewise C^1 topology on the fixed partition $\mathcal{P}$. Let $T^{(n)}=\{\tau_1,\dots,\tau_K;p_1^{(n)},\dots,p_K^{(n)}\}$, $n=1,2,\dots$ be a sequence of random maps. For*

m large, $T^{(n)}$ has an invariant density $f^{(n)}$ and the sequence $\{f^{(n)}\}_{n=1}^{\infty}$ is precompact in L^1. Moreover, any limit point f^* of this sequence is a fixed point of P_T.

Proof. The main part of the proof is to establish an inequality similar to inequality (3.3) uniformly for all n larger than some integer N_1. As a result of applying theorem 3.4 to $T^{(n)}$ we obtain:

$$V(P_{T^{(n)}}f) \le \underbrace{\sigma^{(n)}(1+1/a^{(n)})}_{A^{(n)}} V(f) + \underbrace{(M^{(n)} + \frac{\sigma^{(n)}}{a^{(n)}\delta})}_{B^{(n)}} \|f\|_1, \tag{3.4}$$

where $a^{(n)} = \min\{a^{(n)}(S_i) : i = 1, \dots, q\} > 0$, $\delta^{(n)} = \min\{\delta(S_i), : i = 1, \dots, q\} > 0$, $M_{k,i}^{(n)} = \sup_{x\in S_i}(Dp_k^{(n)}(x) - \frac{DJ_{k,i}}{J_{k,i}}p_k^{(n)}(x))$ and $M^{(n)} = \sum_{k=1}^{K} \max_{1\le i\le q} M_{k,i}^{(n)}$.

Note that $a^{(n)}$ and $\delta^{(n)}$ do not depend on probabilities, so the superscript (n) can be dropped. In order to show that inequality (3.4) holds uniformly it suffices to choose N_1 large enough that $\sigma^{(n)}(1+1/a) < 1$ for all $n > N_1$. This is easily achievable since $p_k^{(n)} \to p_k$ for all $k = 1, \dots, K$. The uniform boundedness of $\left\{f^{(n)}\right\}$ in BV and the invariance of its limit points under P_T follow in a similar way to the one-dimensional case. Note that in this case it is not necessary to consider a higher power of the map $T^{(n)}$ as opposed to the one-dimensional case. □

4. Main Result

Let $T = \{\tau_1, \tau_2; p_1, p_2\}$ be an N-dimensional random map with position dependent probabilities $p_1(x) = 1$ and $p_2(x) = 0$ satisfying conditions of the previous section. Note that T is essentially the same map as τ_1. Suppose the domain of T is $I = I_1 \cup I_2$, where I_1 and I_2 are invariant under τ_1. Suppose τ_1 has exactly two ergodic measures μ_1, and μ_2 with densities f_1 and f_2 on I_1 and I_2, respectively. The map τ_2 differs from τ_1 on the sets $H_{1,2} \subset I_1$ and $H_{2,1} \subset I_2$, where $H_{1,2} = I_1 \cap \tau_2^{-1}(I_2)$ and $H_{2,1} = I_2 \cap \tau_2^{-1}(I_1)$. We assume that

$$\mu_1(H_{1,2}) > 0 \quad \text{and} \quad \mu_2(H_{2,1}) > 0\ . \tag{4.1}$$

Now consider a sequence of random maps $T^{(n)} = \left\{\tau_1, \tau_2; p_1^{(n)}, p_2^{(n)}\right\}$ as perturbations of T, where only the probabilities are changed. Let

$$p_2^{(n)} = p_{2,1}^{(n)}\chi_{H_{2,1}} + p_{1,2}^{(n)}\chi_{H_{1,2}} \tag{4.2}$$

$$p_1^{(n)} = 1 - p_2^{(n)}, \tag{4.3}$$

with $p_{1,2}^{(n)}$, $p_{2,1}^{(n)} > 0$, independent of x. Our main result is the following theorem.

Theorem 4.1. *If $p_{1,2}^{(n)}, p_{2,1}^{(n)} \to 0$ and $\lim_{n\to\infty} \frac{p_{2,1}^{(n)}}{p_{1,2}^{(n)}}$ exists, then the acim's of the n-dimensional random maps $T^{(n)}$ converge to the measure $\mu = \alpha_1\mu_1 + \alpha_2\mu_2$, where*

$$\frac{\alpha_1}{\alpha_2} = \frac{\mu_2(H_{2,1})}{\mu_1(H_{1,2})} \lim_{n\to\infty} \frac{p_{2,1}^{(n)}}{p_{1,2}^{(n)}}\ .$$

Proof. Let $\mu_{T^{(n)}}$ be an acim of $T^{(n)}$ (we do not assume its uniqueness). Let $f^{(n)}$ be the invariant density of $\mu_{T^{(n)}}$.

By Theorem 3.6, $\{f^{(n)}\}_{n\geq 1}$ is precompact in L^1 and if f^* is a limit point, then f^* is of the form $\alpha_1 f_1 + \alpha_2 f_2$ for some $0 \leq \alpha_1, \alpha_2 \leq 1$, $\alpha_1 + \alpha_2 = 1$. In terms of the corresponding measures, there exists a subsequence n_k such that:

$$\mu_{T^{(n_k)}}(H_{1,2}) \to \alpha_1\mu_1(H_{1,2}) + \alpha_2\mu_1(H_{2,1}) = \alpha_1\mu_1(H_{1,2}) \tag{4.4}$$

$$\mu_{T^{(n_k)}}(H_{2,1}) \to \alpha_1\mu_2(H_{1,2}) + \alpha_2\mu_2(H_{2,1}) = \alpha_2\mu_2(H_{2,1}) \tag{4.5}$$

By (4.1), (4.4) and (4.5) we have $\mu_{T^{(n)}}(H_{1,2}) > 0$ or $\mu_{T^{(n)}}(H_{2,1}) > 0$. Let us assume $\mu_{T^{(n)}}(H_{2,1}) > 0$ without loss of generality. Then,

$$\begin{aligned}\mu_{T^{(n)}}(I_1) = \int_I \mathbb{P}(x, I_1) d\mu_{T^{(n)}} &= 1 \cdot \mu_{T^{(n)}}(I_1 \setminus H_{1,2}) + (1 - p_{1,2}^{(n)}) \cdot \mu_{T^{(n)}}(H_{1,2}) \\ &+ 0 \cdot \mu_{T^{(n)}}(I_2 \setminus H_{2,1}) + p_{1,2}^{(n)}\mu_{T^{(n)}}(H_{2,1}).\end{aligned}$$

Hence,

$$\frac{\mu_{T^{(n)}}(H_{1,2})}{\mu_{T^{(n)}}(H_{2,1})} = \frac{p_{2,1}^{(n)}}{p_{1,2}^{(n)}}. \tag{4.6}$$

Applying (4.6), (4.4) and (4.5), we get

$$\frac{\alpha_1}{\alpha_2} = \frac{\mu_2(H_{2,1})}{\mu_1(H_{1,2})} \lim_{k\to\infty} \frac{p_{2,1}^{(n_k)}}{p_{1,2}^{(n_k)}}.$$

□

Additional information about the spectrum of operators $P_{T^{(n)}}$ is provided in the following theorem based on results of [8].

Theorem 4.2. *Let us assume that 1 is an eigenvalue of P_T of multiplicity 2. For arbitrarily small $\delta > 0$, there exists an n_δ such that for $n \geq n_\delta$ the spectrum of $P_{T^{(n)}}$ intersected with $\{z : |z - 1| < \delta\}$ consists of two eigenvalues of multiplicity 1: 1 and r_n, $|r_n| \leq 1$, $r_n \neq 1$ and $r_n \to 1$, as $n \to \infty$.*

Proof. The family $P_{T^{(n)}}$, $n \geq 1$, satisfies the assumptions of Corollary 1 of [8] which implies the above statement. □

5. Main theorem with L ergodic components

Let $T = \{\tau_1, \tau_2; p_1, p_2\}$ be an N-dimensional random map with position dependent probabilities $p_1(x) = 1$ and $p_2(x) = 0$. So T is essentially the same map as τ_1. Suppose τ_1 has L ergodic components $I_1, \ldots, I_L$, $\cup_{i=1}^L I_i = I$. Suppose there are $L-1$ pairwise disjoint "holes" $\{H_{i,j}\}_{1\leq j\leq L,\ j\neq i}$ in each component I_i. Map τ_2 is defined as a piecewise expanding map which has the following properties

$$\tau_2(H_{i,j}) \subset I_j \ , \text{ for } \ i, j \in \{1, \ldots, L\}$$

and $\tau_2 = \tau_1$ outside the holes.

Let $T^{(n)} = \{\tau_1, \tau_2; p_1^{(n)}, p_2^{(n)}\}$ be a sequence of random maps such that

$$1 - p_1^{(n)} = p_2^{(n)} = \sum_{i=1}^{L} \sum_{j\neq i} p_{i,j}^{(n)} \chi_{H_{i,j}}, \tag{5.1}$$

$0 < p_{i,j}^{(n)} < 1$ and

$$p_{i,j}^{(n)} = h(n)a_{i,j} + o(h(n)) \ , \tag{5.2}$$

for some function h such that $\lim_{n\to\infty} h(n) = 0$. Let $\mu_T^{(n)}$ denote the invariant measure of $T^{(n)}$. Then for every $1 \le k \le L$,

$$\mu_{T^{(n)}}(I_k) = \int \mathbb{P}(x, I_k) d\mu_{T^{(n)}}(x) = \int_{\tau_1^{-1}(I_k)} p_1^{(n)}(x) d\mu_{T^{(n)}} + \int_{\tau_2^{-1}(I_k)} p_2^{(n)}(x) d\mu_{T^{(n)}} \ .$$

It follows that for every $1 \le k \le L$,

$$\sum_{j\neq k} p_{k,j}^{(n)} \mu_{T^{(n)}}(H_{k,j}) = \sum_{i\neq k} p_{i,k}^{(n)} \mu_{T^{(n)}}(H_{i,k}). \tag{5.3}$$

The left hand side of the equation (5.3) can be interpreted as the amount of $\mu_{T^{(n)}}$-measure that leaves the component I_k and the right hand side as the amount of $\mu_{T^{(n)}}$-measure that enters the component I_k. Intuitively, these two quantities are equal because $\mu_{T^{(n)}}$ is preserved under $T^{(n)}$.

Let us define $q_{i,j} = a_{i,j}\mu_i(H(i,j))$ for $j \neq i$, $q_{i,i} = 1 - \sum_{j\neq i} q_{i,j}$ for $1 \le i \le L$, and

$$Q = [q_{i,j}]_{1\le i,j \le L} \ . \tag{5.4}$$

By the continuity theorem for random maps, there exists a subsequence n_k such that $\mu_{T^{(n_k)}} \to \mu_T = \sum_{i=1}^L \alpha_i \mu_i$. Therefore, $\mu_{T^{(n_k)}}(H_{i,j}) \to \alpha_i\mu_i(H_{i,j})$. Hence, the equations (5.3), for $n = n_k$, can be written as

$$\sum_{j\neq k} a_{k,j}\alpha_k \mu_k(H_{k,j}) = \sum_{i\neq k} a_{i,k}\alpha_i\mu_i(H_{i,k}) + o(1) \ ,$$

or

$$(1 - q_{k,j})\alpha_k = \sum_{i\neq k} q_{i,k}\alpha_i + o(1) \ ,$$

which in matrix form is

$$\alpha Q = \alpha + o(1) \ ,$$

where $\alpha = (\alpha_1, \dots, \alpha_L)$. If, for $w = (w_1, \dots, w_L)$ the solution of the equation $w = wQ$ is stable under small perturbations, then, $\alpha = (\alpha_1, \dots, \alpha_L)$ satisfies

$$\alpha \cdot Q = \alpha.$$

The conditions for stability of eigenvectors for probability matrices are well known, see for example [3].

We proved the following theorem

Theorem 5.1. *Let $T^{(n)}$ be a sequence of random maps satisfying assumptions of Section 4 but such that map τ_1 has $L \ge 2$ ergodic components. Let probabilities $p_{i,j}^{(n)}$, $1 \le i,j \le L$ satisfy assumptions (5.2). If the matrix Q defined in (5.4) has stable left 1-eigenvector, then the invariant measures $\mu_{T^{(n)}}$ converge as $n \to \infty$ to the measure $\sum_{i=1}^L \alpha_i\mu_i$, where $\alpha Q = \alpha$, and μ_i is the τ_1-invariant measure on the i-th ergodic component.*

6. Examples

Example 6.1.

We now present a simple Markov map example on the interval [0,1]. Consider the maps τ_1 and τ_2 as shown in figure 1.

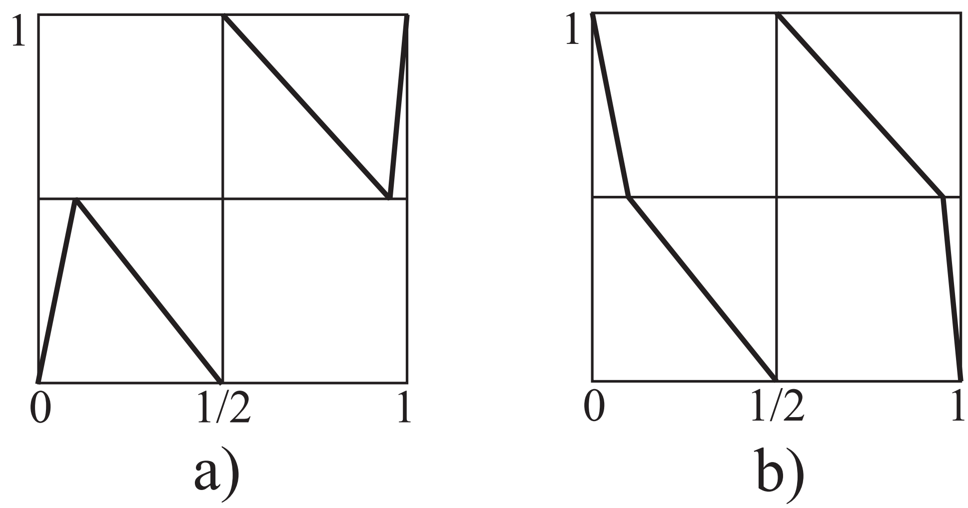

Figure 1. Maps τ_1 and τ_2.

Both maps are Markov on the partition $\mathcal{P} = \{J_1 = [0, 0.1], J_2 = [0.1, 0.5], J_3 = [0.5, 0.95], J_4 = [0.95, 1]\}$. Let $|J|$ denote the Lebesgue measure of the set J. Then $|J_1| = 0.1$, $|J_2| = 0.4$, $|J_3| = 0.45$, $|J_4| = 0.05$. τ_1 and τ_2 have slopes of the same magnitude on $J_1, \ldots, J_4$. They are $s_1 = 5$, $s_2 = 5/4$, $s_3 = 10/9$, $s_4 = 10$, respectively. The ergodic components of τ_1 are $I_1 = J_1 \cup J_2$ and $I_2 = J_3 \cup J_4$. The holes are $H_{1,2} = J_1$ and $H_{2,1} = J_4$.

Our aim is to compute the acims of the random maps $T = \{\tau_1, \tau_2, 1, 0\}$ and $T^{(n)} = \{\tau_1, \tau_2, p_1^{(n)}, p_2^{(n)}\}$, where $p_1^{(n)}$ and $p_2^{(n)}$ are defined as in equation (4.3) and (4.2). To this end, we will first compute the invariant densities of τ_1 and τ_2.

The matrices corresponding to Perron-Frobenius operators for τ_1 and τ_2 are

$$M_{\tau_1} = \begin{bmatrix} 1/5 & 1/5 & 0 & 0 \\ 4/5 & 4/5 & 0 & 0 \\ 0 & 0 & 9/10 & 9/10 \\ 0 & 0 & 1/10 & 1/10 \end{bmatrix} \quad , \quad M_{\tau_2} = \begin{bmatrix} 0 & 0 & 1/5 & 1/5 \\ 4/5 & 4/5 & 0 & 0 \\ 0 & 0 & 9/10 & 9/10 \\ 1/10 & 1/10 & 0 & 0 \end{bmatrix} .$$

Any invariant density of τ_1 or τ_2 is piecewise constant on the partition $\mathcal{P}$. Moreover, if we denote the value of the invariant density on J_i by f_i, $1 \le i \le 4$, then (f_1, f_2, f_3, f_4) is the left eigenvector of the Perron-Frobenius matrix corresponding to eigenvalue 1. For τ_2, one easily checks that $(2/3, 2/3, 4/3, 4/3)$ is the unique normalized invariant density. On the other hand, τ_1 has two ergodic components with acims μ_1 and μ_2 which are simply the normalized Lebesgue measure on I_1 and I_2, respectively. Any acim of τ_1 is of the form $t\mu_1 + (1-t)\mu_2$, $0 \le t \le 1$.

It follows from equation (2.6) that the invariant density of T is the same as the invariant density of τ_1.

For the random map $T^{(n)} = \{\tau_1, \tau_2; p_1^{(n)}, p_2^{(n)}\}$, equation (2.6) implies that the invariant density $(f_1^{(n)}, f_2^{(n)}, f_3^{(n)}, f_4^{(n)})$ satisfies

$$\begin{aligned}\left(f_1^{(n)}, f_2^{(n)}, f_3^{(n)}, f_4^{(n)}\right) &= \left((1-p_{1,2}^{(n)})f_1, f_2^{(n)}, f_3^{(n)}, (1-p_{2,1}^{(n)})f_4^{(n)}\right) M_{\tau_1} \\ &+ \left(p_{1,2}^{(n)} f_1^{(n)}, 0, 0, p_{2,1}^{(n)} f_4^{(n)}\right) M_{\tau_2},\end{aligned} \tag{6.1}$$

which yields $f_1^{(n)} = f_2^{(n)}$, $f_3^{(n)} = f_4^{(n)}$ and $p_{2,1}^{(n)} f_4^{(n)} = 2p_{1,2}^{(n)} f_1^{(n)}$. So the unique normalized invariant density for $T^{(n)}$ is

$$f^{(n)} = \frac{2}{p_{2,1}^{(n)} + 2p_{1,2}^{(n)}} \left(p_{2,1}^{(n)}, p_{2,1}^{(n)}, 2p_{1,2}^{(n)}, 2p_{1,2}^{(n)}\right).$$

Suppose $\lim_{n\to\infty} p_{2,1}^{(n)}/p_{1,2}^{(n)} = l$. Then $f^{(n)} \to (2/(2+l))(l,l,2,2)$. It follows that the invariant measure $\mu_{T^{(n)}} \to \alpha_1\mu_1 + \alpha_2\mu_2$, where $\alpha_1 = (2l)/(l+2)$ and $\alpha_2 = 4/(l+2)$. Moreover,

$$\frac{\alpha_1}{\alpha_2} = \frac{1}{2} l = \frac{0.05}{0.1} l = \frac{\mu_2(H_{2,1})}{\mu_1(H_{1,2})} \lim_{n\to\infty} \frac{p_{2,1}^{(n)}}{p_{1,2}^{(n)}}.$$

The Perron-Frobenius operator for the random map $T^{(n)}$ corresponds to the matrix (already shown in (6.1))

$$M_{T^{(n)}} = \begin{bmatrix} 1/5-(1/5)p_{1,2}^{(n)} & 4/5 & 0 & (1/10)p_{1,2}^{(n)} \\ 1/5 & 4/5 & 0 & 0 \\ 0 & 0 & 9/10 & 1/10 \\ (1/5)p_{2,1}^{(n)} & 0 & 9/10 & 1/10-(1/10)p_{2,1}^{(n)} \end{bmatrix},$$

with eigenvalues: 1, $r_1^{(n)} = 1/2 - (1/20)p_{2,1}^{(n)} - (1/10)p_{1,2}^{(n)} + a$, $r_2^{(n)} = 1/2 - (1/20)p_{2,1}^{(n)} - (1/10)p_{1,2}^{(n)} - a^{(n)}$ and 0, where

$$a^{(n)} = (1/20)\sqrt{100 + 16p_{2,1}^{(n)} + 24p_{1,2}^{(n)} + (p_{2,1}^{(n)})^2 + 4p_{2,1}^{(n)}p_{1,2}^{(n)} + 4(p_{1,2}^{(n)})^2}\,.$$

For $p_{1,2}^{(n)}$ and $p_{2,1}^{(n)}$ close to 0, we have $r_1^{(n)}$ close to 1 and $r_2^{(n)}$ close to 0. For example, if $p_{1,2}^{(n)} = p_{2,1}^{(n)} = 0.01$, then $r_1^{(n)} \sim 0.9995$ and $r_2^{(n)} \sim -0.0025$. The eigenvector corresponding to $r_1^{(n)}$ is $v \sim [-0.749265, -0.751139, 0.375571, 0.373698]$.

Example 6.2.

We present a random map with 3 ergodic components of the original map τ_1, see figure 2. Consider maps τ_1 and τ_2 on a set $I = [0,1]$: τ_1 has three ergodic components $I_1 = [0,1/3]$, $I_2 = [1/3,2/3]$ and $I_3 = [2/3,1]$, $\cup_{i=1,2,3} I_i = I$. On each components normalized Lebesgue measure μ_i, $i = 1,2,3$, is τ_1-invariant. There are 2 holes in each component. They are

$$H_{1,2} = [1/9, 2/9], H_{1,3} = [2/9, 1/3] \subset I_1\ ;$$
$$H_{2,1} = [1/3, 4/9], H_{2,3} = [5/9, 2/3] \subset I_2\ ;$$
$$H_{3,1} = [2/3, 7/9], H_{3,2} = [7/9, 8/9] \subset I_3\ .$$

Map τ_2 is defined as a piecewise expanding map shown in Fig. 2. It has the following properties

$$\tau_2(H_{i,j}) \subset I_j\ ,\ \text{for}\ \ i,j \in \{1,2,3\}$$

and $\tau_2 = \tau_1$ outside the holes.

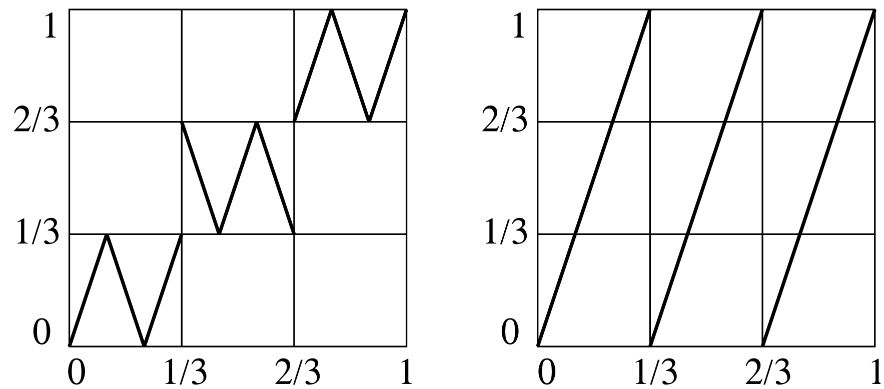

FIGURE 2. Maps τ_1 and τ_2 for Example 6.2 with 3 ergodic components.

We define the probabilities that each of the holes will be used by

$$p_{i,j}^{(n)} = h(n)a_{i,j} + o(h(n)), \quad 1 \le i,j \le 3 \ ,$$

where h is such that $\lim_{n\to\infty} h(n) = 0$ and the matrix $A = [a_{i,j}]_{1\le i,j\le 3}$ is given by

$$A = \begin{bmatrix} 0 & 0.3 & 0.5 \\ 0.7 & 0 & 0.2 \\ 0.1 & 0.1 & 0 \end{bmatrix} .$$

The position dependent probability of applying the map τ_2 is defined by

$$p_2^{(n)}(x) = \sum_{i=1,2,3} \sum_{j\ne i} p_{i,j}^{(n)} \chi_{H_{i,j}}(x) \ , \quad x \in I. \tag{6.2}$$

The probability of applying map τ_1 is defined by $p_1^{(n)}(x) = 1 - p_2^{(n)}(x)$, $x \in I$.

Consider the random map $T^{(n)} = \{\tau_1, \tau_2; p_1^{(n)}, p_2^{(n)}\}$. Let $\mu_{T^{(n)}}$ be its invariant measure. By the continuity theorem, $\mu_{T^{(n)}} \to \alpha_1\mu_1 + \alpha_2\mu_2 + \alpha_3\mu_3$ as $p_{i,j} \to 0$, $i \ne j$. Since $\mu_i(H_{i,j}) = 1/3$ for $i \ne j$, $1 \le i,j \le 3$, by (5.4) we have

$$Q = \frac{1}{30}\begin{bmatrix} 22 & 3 & 5 \\ 7 & 21 & 2 \\ 1 & 1 & 28 \end{bmatrix} .$$

Therefore, the normalized vector $\alpha = (\alpha_1, \alpha_2, \alpha_3) = \frac{1}{78}(16, 11, 51)$.

Example 6.3.

We consider a two dimensional Markov map example with τ_1 having 4 ergodic components. A Maple 12 program with the details of the example is available on request. We will use the notation of Example 6.2. The space I is a unit square of the plane $\mathbb{R}^2$. It is divided into 4 identical subsquares I_1, I_2, I_3, I_4 and each of them is further divided into 9 identical smaller subsquares: $I_1 = \cup_{i=1}^{9} S_i$, $I_2 = \cup_{i=10}^{18} S_i$, $I_3 = \cup_{i=19}^{27} S_i$, $I_4 = \cup_{i=28}^{36} S_i$, as in figure 6.

We define τ_1 restricted to each of I_i, $i = 1,2,3,4$, as the same Markov map transforming each square S_j onto four squares S_k in such a way that the corresponding adjacency matrix

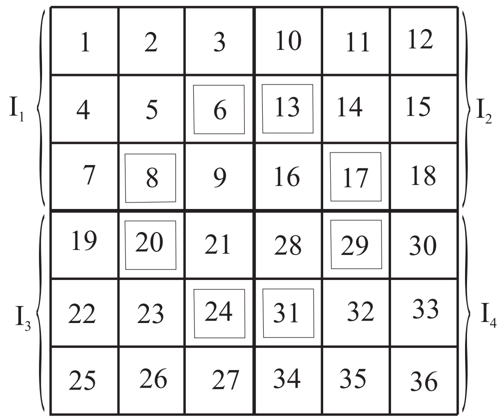

FIGURE 3. The Markov partition for map τ_1 of Example 6.3.

of the map τ_1 restricted to I_i is

$$M = \frac{1}{4}\begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \end{bmatrix} .$$

The matrix M_{τ_1} corresponding to τ_1 is the block matrix with 4 matrices M along the diagonal. The map τ_1 has 4 ergodic components. For each component the normalized acim μ_i , $i = 1, 2, 3, 4$, invariant for τ_1 restricted to I_i, can be represented by the vector

$$\begin{aligned}&[\mu_i(1), \mu_i(2), \mu_i(3), \mu_i(4), \mu_i(5), \mu_i(6), \mu_i(7), \mu_i(8), \mu_i(9)] \\ &= [0.05357, 0.16071, 0.10714, 0.08036, 0.25, 0.16964, 0.02679, 0.08929, 0.0625] .\end{aligned}$$

The squares $S_6, S_8, S_{13}, S_{17}, S_{20}, S_{24}, S_{29}, S_{31}$, are designated as holes. We have $S_6 = H_{1,2}$, $S_8 = H_{1,3}$, $S_{13} = H_{2,1}$, $S_{17} = H_{2,4}$, $S_{20} = H_{3,1}$, $S_{24} = H_{3,4}$, $S_{29} = H_{4,2}$, $S_{31} = H_{4,3}$. We have $\mu_1(S_6) = \mu_3(S_{24}) = 0.16964$, $\mu_1(S_8) = \mu_2(S_{17}) = 0.08929$, $\mu_2(S_{13}) = \mu_4(S_{31}) = 0.08036$, $\mu_3(S_{20}) = \mu_4(S_{29}) = 0.16071$.

We define τ_2 to be the Markov map on I which realizes the transfers. On squares which are not holes it is equal to τ_1. On each of the squares which is a hole τ_2 is a linear map transferring this square onto four squares in appropriate component I_j. The matrix M_{τ_1} has most of its rows the same as the matrix M_{τ_1}, except for rows $6, 8, 13, 17, 20, 24, 29, 31$ which have elements $(6, 10)$, $(6, 11)$, $(6, 13)$, $(6, 14)$, $(8, 19)$, $(8, 20)$, $(8, 22)$, $(8, 23)$, $(13, 5)$, $(13, 6)$, $(13, 8)$, $(13, 9)$, $(17, 29)$, $(17, 30)$, $(17, 32)$, $(17, 33)$, $(20, 4)$, $(20, 5)$, $(20, 7)$, $(20, 8)$, $(24, 31)$, $(24, 32)$, $(24, 34)$, $(24, 35)$, $(29, 14)$, $(29, 15)$, $(29, 17)$, $(29, 18)$, $(31, 20)$, $(31, 21)$, $(31, 23)$, $(31, 24)$, equal to $1/4$ and all other elements 0.

Let h be such that $\lim_{n\to\infty} h(n)=0$. We define the matrix of transfer probabilities between I_i and I_j as

$$P^{(n)}=\left[p_{i,j}^{(n)}\right]_{1\le i,j\le 4}=h(n)\cdot A \quad , \quad \text{where} \quad A=\begin{bmatrix} 0 & 0.4 & 0.5 & 0 \\ 0.3 & 0 & 0 & 0.8 \\ 0.7 & 0 & 0 & 0.5 \\ 0 & 0.6 & 0.6 & 0 \end{bmatrix} .$$

We define position dependent probabilities $p_1^{(n)}, p_2^{(n)}$ as in (5.1). The random map $T^{(n)}=\{\tau_1,\tau_2;p_1^{(n)},p_2^{(n)}\}$ has matrix $M_{T^{(n)}}$ with rows the same as the rows of M_{τ_1} except for rows $6,8,13,17,20,24,29,31$ defined by

$$\begin{aligned}
\text{row}(6,M_{T^{(n)}}) &= (1-p_{1,2}^{(n)})\text{row}(6,M_{\tau_1})+p_{1,2}^{(n)}\cdot\text{row}(6,M_{\tau_2}) \ ,\\
\text{row}(8,M_{T^{(n)}}) &= (1-p_{1,3}^{(n)})\text{row}(8,M_{\tau_1})+p_{1,3}^{(n)}\cdot\text{row}(8,M_{\tau_2}) \ ,\\
\text{row}(13,M_{T^{(n)}}) &= (1-p_{2,1}^{(n)})\text{row}(13,M_{\tau_1})+p_{2,1}^{(n)}\cdot\text{row}(13,M_{\tau_2}) \ ,\\
\text{row}(17,M_{T^{(n)}}) &= (1-p_{2,4}^{(n)})\text{row}(17,M_{\tau_1})+p_{2,4}^{(n)}\cdot\text{row}(17,M_{\tau_2}) \ ,\\
\text{row}(20,M_{T^{(n)}}) &= (1-p_{3,1}^{(n)})\text{row}(20,M_{\tau_1})+p_{3,1}^{(n)}\cdot\text{row}(20,M_{\tau_2}) \ ,\\
\text{row}(24,M_{T^{(n)}}) &= (1-p_{3,4}^{(n)})\text{row}(24,M_{\tau_1})+p_{3,4}^{(n)}\cdot\text{row}(24,M_{\tau_2}) \ ,\\
\text{row}(29,M_{T^{(n)}}) &= (1-p_{4,2}^{(n)})\text{row}(29,M_{\tau_1})+p_{4,2}^{(n)}\cdot\text{row}(29,M_{\tau_2}) \ ,\\
\text{row}(31,M_{T^{(n)}}) &= (1-p_{4,3}^{(n)})\text{row}(31,M_{\tau_1})+p_{4,3}^{(n)}\cdot\text{row}(31,M_{\tau_2}) \ .
\end{aligned}$$

The $T^{(n)}$-invariant measure $\mu_{T^{(n)}}$ has been obtained using Maple. We define the vector $\alpha^{(n)}=[\mu_{T^{(n)}}(I_1),\mu_{T^{(n)}}(I_2),\mu_{T^{(n)}}(I_3),\mu_{T^{(n)}}(I_3)]$. Then,

$$\alpha^{(n)}=\frac{1}{126509}\,[25416,52668,14130,34295]+O(h(n)) \ .$$

The matrix Q is defined as in (5.4). The left 1-eigenvector of Q is equal to $\lim_{n\to\infty}\alpha^{(n)}$.

For $\epsilon:=h(n)$ close to 0 the matrix corresponding to Frobenius–Perron operator of $T^{(n)}$ has, except 1, three other eigenvalues close to 1 but different from 1. For $\epsilon=10^{-3}$ they are 0.9997176900 , 0.9998399077 and 0.9998924535. For $\epsilon=10^{-4}$ we obtained 0.9999717673, 0.9999839914, 0.9999892419.

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